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Towards Transport Theory of Hadron Gases

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An important role of hadron resonances for determining the characteristics of hadron gases is argued. A kinetic theory model of hadron gas is developed. A classical, nonquantum, distribution function of a resonance is defined with the help of the profile function being an analogue of the mass shell delta function of stable particles. The Boltzmann equation is generalized to include the resonance decay and resonance formation processes. To determine the unknown profile function, the transition rates are assumed to satisfy the bilateral normalization or the detailed balance condition. The profile function is expressed through the resonance formation cross section and the decay width. The H-theorem is proved, and it is shown that the form of the equilibrium distribution function of a resonance coincides with that of a stable particle. Macroscopic equilibrium characteristics are studied. Significance of the resonance mass smearing effect is demonstrated.

I. INTRODUCTION

Considering hadron gases, one finds that the standard kinetic theory, see, e.g., [1], developed for the description of atomic gases is far inadequate for studying hadrons. The majority of hadrons are unstable. The lifetime of hadron resonances is so short that the decay width is comparable to the particle mass. The resonances are abundantly produced in hadron–hadron collisions, see, e.g., [2]. Thus, to describe a hadron gas, the resonances have to be taken into account. However, for the inclusion of unstable particles, the Boltzmann equation has to be modified.

According to the current understanding of a hadron structure, hadrons are built of quarks and gluons. So, the transport theory of hadrons should be constructed on the basis of quantum chromodynamics. However, it is not possible to realize this program now because of the unsolved problems of QCD concerning the hadron structure and the inter-hadron forces. In the absence of underlying microscopic dynamics one can develop phenomenological models based on physical arguments. In this paper we present such an attempt. We consider a minimal modification of

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the standard classical (non-quantum) kinetic theory [1] keeping in mind all limitations of this theory like the absence of two-particle correlations, etc. In our model, besides binary collisions, we take into account two-particle resonance decays and time-reversed processes, i.e., resonance formation. Then, we explicitly include the effect of resonance mass smearing.

There are other characteristics features of hadrons which are still outside our discussion. Of particular importance is, in our opinion, the fact that many hadrons may be produced in hadron collisions. Binary collisions dominate at relatively low incident energies only. The inclusion of such processes is difficult and resembles the problems found in attempts to develop the transport theory of dense atomic gases, see, e.g., [1].

The non-trivial equilibrium properties of a hadron gas have been widely discussed in numerous papers by R. Hagedron and collaborators [3, 4]. The Gibbs statistical mechanics and the idea of statistical bootstrap have been used in their considerations. The most important, in our opinion, of Hagedorn's results is the prediction of limiting hadron temperature contemporarily interpreted as a temperature of phase transition to quark–gluon plasma [4]. Because this experimentally confirmed temperature is close to the pion mass, the average incident energy of hadron collisions in the gas in equilibrium does not exceed some hundreds of MeV. In this incident energy region binary collisions dominate, which makes our considerations (limited to decays, resonance formation, and binary collisions) applicable for the description of hadron gas close to equilibrium. Anyhow our discussion is adequate for dilute gases, where one can neglect the collisions with more than two particles in an initial state.

Our paper is organized as follows. In Section II we define the classical distribution function of resonance and some macroscopical quantities. The Boltzmann equation is generalized in Section III. In Section IV the H-theorem is proved and an equilibrium state is considered. The equilibrium characteristics of a hadron gas are discussed in Section V. In Section VI we conclude our considerations.

II. The Distribution Function of a Resonance

In our consideration we follow the textbook "Relativistic Kinetic Theory" by de Groot, van Leeuwen, and van Weert [5]. Because the energy, E, and momentum, $\mathbf{p}$, of a resonance are not connected by the mass relation

$$E^2 - \mathbf{p}^2 = m^2 \qquad (c = k = \hbar = 1)$$

the four-dimensional, relativistic formalism is a more natural framework for studying hadron gases than the three-dimensional non-relativistic one.

The Lorentz invariant phase-space element of a stable particle d^3p/E is not adequate for a resonance since the energy and momentum have to be independent (quasi-independent) variables. However,

$$\frac{d^3\mathbf{p}}{E} = 2d^4p\delta(p^2 - m^2)\,\Theta(E), \tag{1}$$

where $p^2 \equiv p^\mu p_\mu$, $p \equiv p^\mu = (E, \mathbf{p})$. The above expression suggests the form of a resonance phase space element

$$d^4p\,\Delta(p^2), \tag{2}$$

where the function Δ, later on called the profile function, describes the mass smearing of a resonance. We demand Δ to be a Lorentz scalar.

The profile function is assumed to depend on p^2 while it is independent of any gas characteristics. In particular, we assume that a particle lifetime does not depend on the gas density. In general it is not true because the density of final states of a decay process can be significantly different in vacuum and in a dense gas at low temperature. For example, due to the Pauli quenching, the lifetime of the N^* resonance decaying into a pion and a nucleon can be much longer in nuclear matter than in vacuum. We conclude as follows. Assuming that the profile function depends on p^2 only, we limit our considerations to classical gases. The form of the $\Delta(p^2)$ function will be discussed in the next section, where the connection with experimentally measurable quantities will be established.

We define the distribution function so that

$$f(p, x)\, d^3\mathbf{x}\, d^4p\, \Delta(p^2)E, \qquad x \equiv (t, \mathbf{x}) \tag{3}$$

gives an average number of resonances being at a moment of time t in the space element $d^3\mathbf{x}$ with the four-momentum between p and $p + d^4p$. The above definition will be more obvious if we write down the particle four-flow vector

$$N^\mu(x) = \int d^4p\, \Delta(p^2)\, p^\mu f(p, x), \tag{4}$$

which is an analogue of a stable particle four-flow

$$N^\mu_{\mathrm{st}}(x) = \int \frac{d^3\mathbf{p}}{E}\, p^\mu f(p, x).$$

The definition (3) plays a crucial role in our considerations and lets us employ the standard scheme of the kinetic theory for studying hadron resonances.

Dealing with decaying particles, we are forced to consider a mixture of many sorts of particles. Thus, we denote by $f_i(x, p)$ the distribution function of an ith sort of particles. The energy-momentum tensor and the entropy four-flow are the following

$$T^{\mu\nu}(x) = \sum_i \int d^4\tilde{p}_i \, p^\mu p^\nu f_i(p, x),$$
$$S^\mu(x) = \sum_i \int d^4\tilde{p}_i \, p^\mu f_i(p, x)[\ln f_i(p, x) - 1], \tag{5}$$

$d^4\tilde{p}$ is the phase-space element of a stable particle (1) or a resonance (2).

III. The Kinetic Equations

Let us consider the mixture of N^s and N^u sorts of stable and unstable particles, respectively. Assuming that a resonance decays into two stable particles, one finds the following set of kinetic equations:

$$p^\mu \, \partial_\mu f_i(p, x) = C_i^s + D_i^s, \qquad i = 1, 2, ..., N^s \tag{6a}$$

and

$$p^\mu \, \partial_\mu f_j(p, x) = C_j^u + D_j^u, \qquad j = 1, 2, ..., N^u, \tag{6b}$$

where C_i is the standard collision terms describing the binary interactions, see, e.g., [5]. When a resonance is involved in a collision, the phase-space element (1) has to replaced by (2).

$$D_i^s \equiv \sum_{j=1}^{N^u} \sum_{k=1}^{N^s} \int d^4p_j \, \Delta^j(p_j^2) \frac{d^3\mathbf{p}_k}{E_k},$$
$$\cdot \, [f_j(p_j, x) \, W^{j \to ik}(p_j \mid p, p_k) - f_i(p, x) \, f_k(p_k, x) \, W^{ik \to j}(p, p_k \mid p_j)],$$
$$D_j^u \equiv \sum_{i=1}^{N^s} \sum_{k=1}^{N^s} \int \frac{d^3\mathbf{p}_i}{E_i} \frac{d^3\mathbf{p}_k}{E_k}$$
$$\cdot \, [f_i(p_i, x) \, f_k(p_k, x) \, W^{ik \to j}(p_i, p_k \mid p) - f_j(p, x) \, W^{j \to ik}(p \mid p_i, p_k)],$$

where $W^{j \to ik}(p_j \mid p_i, p_k)$ is the transition rate for the decay of the resonance of a jth sort having four-momentum p_j into two particles of ith and kth sorts with momenta p_i and p_k. $W^{ik \to j}(p_i, p_k \mid p_j)$ is the transition rate of the inverse process of resonance formation.

Let us rewrite Eq. (6b) in the non-covariant, more familiar form

$$\frac{\partial}{\partial t} f_j(p, x) + \mathbf{V}\nabla f_j(p, x)$$
$$= \frac{C_j^{\mathrm{u}}}{E} + \sum_{i=1}^{N^{\mathrm{s}}} \sum_{k=1}^{N^{\mathrm{s}}} \int d^3p_i\, d^3p_k$$
$$\cdot \left[f_i(p_i, x) f_k(p_k, x) \frac{W^{ik \to j}(p_i, p_k \mid p)}{EE_iE_k} - f_j(p, x) \frac{W^{j \to ik}(p \mid p_i, p_k)}{EE_iE_k} \right], \tag{7}$$

where $\mathbf{V} \equiv \mathbf{p}/E$. Recalling a physical interpretation of the distribution function, one finds from (7) the following connection of the transition rates with the measurable quantities

$$\frac{(2\pi)^3}{|\mathbf{V}_i - \mathbf{V}_k|} \frac{W^{ik \to j}(p_i, p_k \mid p_j)}{E_iE_jE_k} d^4p_j\, \Delta^j(p_j^2)\, E_j = d\sigma^{ik \to j} \tag{8}$$

and

$$\frac{W^{j \to ik}(p_j \mid p_i, p_k)}{E_iE_jE_k} d^3\mathbf{p}_i\, d^3\mathbf{p}_k = d\Gamma^{j \to ik}, \tag{9}$$

where $\sigma^{ik \to j}$ is the cross section of jth resonance formation, $|\mathbf{V}_i - \mathbf{V}_k|$ is the relative velocity of particles with four-momenta p_i and p_k, and $\Gamma^{j \to ik}$ is the partial decay width. The presence of a $(2\pi)^3$ coefficient in formula (8) is related to the fact that the phase-space elements present in the kinetic equations are not divided by $(2\pi)^3$ while in the units which are used $\hbar$ equals unity. Thus, the $(2\pi)^3$ coefficients are absorbed by the transition rates.

Since a four-momentum is conserved in any reaction, one can write

$$W^{j \to ik}(p_j \mid p_i, p_k) = a(p_j \mid p_i p_k)\, \delta^{(4)}(p_j - p_i - p_k), \tag{10}$$

$$W^{ik \to j}(p_i, p_k \mid p_j) = a(p_i p_k \mid p_j)\, \delta^{(4)}(p_j - p_i - p_k). \tag{11}$$

Substituting (10) and (11) in (8) and (9), we determine the coefficients a. For the decay process the decay products are assumed to be isotropically distributed in the center-of-mass of decaying particle. In this way we arrive at the formulae

$$W^{ik \to j}(p_i, p_k \mid p_j) = \frac{F_{ik}\sigma^{ik \to j}}{(2\pi)^3 \Delta^j(p_j^2)} \delta^{(4)}(p_j - p_i - p_k), \tag{12}$$

$$W^{j \to ik}(p_j \mid p_i, p_k) = \frac{M_j \Gamma^{j \to ik}}{L_{ik}} \delta^{(4)}(p_j - p_i - p_k), \tag{13}$$

where F_{ik} is the Lorentz invariant flux factor

$$F_{ik} \equiv E_i E_k |\mathbf{V}_i - \mathbf{V}_k| = ((p_i p_k)^2 - p_i^2 p_k^2)^{1/2} = \tfrac{1}{2}((M_j^2 - m_j^2 - m_k^2)^2 - 4m_i^2 m_k^2)^{1/2},$$
$$M_j^2 \equiv (p_i + p_k)^2 = p_j^2.$$

L_{ik} is the Lorentz invariant two-particle phase-space

$$L_{ik} \equiv \int \frac{d^3\mathbf{p}_i}{E_i} \frac{d^3\mathbf{p}_k}{E_k} \delta^{(4)}(p_j - p_i - p_k) = \frac{2\pi}{M_j^2} ((M_j^2 - m_i^2 - m_k^2)^2 - 4m_i^2 m_k^2)^{1/2}.$$

L and F are related by the formula $L_{ik} = (4\pi/M_j^2)\, F_{ik}$.

Let us discuss how to determine the profile function. If we assume that the transition rates satisfy the detailed balance condition

$$W^{j \to ik}(p_j | p_i, p_k) = W^{ik \to j}(p_i, p_k | p_j)$$

one gets

$$\Delta^j(p_j^2) = \frac{1}{(2\pi)^3} \frac{L_{ik} F_{ik}}{M_j} \frac{\sigma^{ik \to j}}{\Gamma^{j \to ik}}. \tag{14}$$

Below we will discuss the above formula. But now we show another way leading to Eq. (14). We assume that the transition rates satisfy the bilateral normalization conditions

$$\sum_j \int d^4p_j\, \Delta^j(p_j^2)\, W^{ik \to j}(p_i, p_k | p_j) = \sum_j \int d^4p_j\, \Delta^j(p_j^2)\, W^{j \to ik}(p_j | p_i, p_k) \tag{15}$$

and

$$\sum_{i,k} \int \frac{d^3p_i}{E_i} \frac{d^3p_k}{E_k} W^{ik \to j}(p_i, p_k | p_j) = \sum_{i,k} \int \frac{d^3p_i}{E_i} \frac{d^3p_k}{E_k} W^{j \to ik}(p_j | p_i, p_k). \tag{16}$$

The above expressions related to unitarity of the S-matrix are briefly discussed in the Appendix.

Putting (12) and (13) in (15) and (16), one finds the equations

$$\sum_j \frac{F_{ik} \sigma^{ik \to j}}{(2\pi)^3} = \sum_j \frac{M_j \Gamma^{j \to ik}}{L_{ik}} \Delta^j(p_j^2)$$

and

$$\sum_{i,k} \frac{F_{ik} \sigma^{ik \to j} L_{ik}}{(2\pi)^3 \Delta^j(p_j^2)} = \sum_{i,k} M_j \Gamma^{j \to ik}.$$

Because the first equation has to be satisfied for any i, k pairs while the second one for any j, we get the relation

$$\frac{F_{ik}\sigma^{ik\to j}}{(2\pi)^3}=\frac{M_j \Delta^j(p_j^2)\,\Gamma^{j\to ik}}{L_{ik}}$$

which is equivalent to Eq. (14). In this approach formula (14) provides the detailed balance condition. Thus, the detailed balance appeared to be a consequence of the bilateral normalization conditions (15) and (16), the four-momentum conservation, and the assumption of the isotropic distribution of decay products in the center-of-mass of the decaying particle.

As the profile function characterizes a resonance but not a decay channel, formula (14) should give the same result for different i, k pairs. We cannot rigorously prove that the profile function described by (14) is unique. However, we present simplified argumentation and then we show that the independence of i, k indexes is realized for the Breit–Wigner form of the cross section.

One expects the relation

$$\frac{|\mathcal{M}^{ik\to j}|^2}{|\mathcal{M}^{ln\to j}|^2}=\frac{|\mathcal{M}^{j\to ik}|^2}{|\mathcal{M}^{j\to ln}|^2}, \tag{17}$$

where $\mathcal{M}$ is the transition matrix of the indicated process. The above relation is strictly correct when the interaction is invariant under time inversion which is the case, at least approximately, for strong interactions. If the decay width can be factorized as

$$\Gamma^{j\to ik}=|\mathcal{M}^{j\to ik}|^2\,L_{ik}, \tag{18}$$

we get from Eq. (17) the condition

$$\frac{F_{ik}\sigma^{ik\to j}}{F_{ln}\sigma^{ln\to j}}=\frac{\Gamma^{j\to ik}/L_{ik}}{\Gamma^{j\to ln}/L_{ln}},$$

which makes formula (14) independent of i, k indexes. However, the factorization (18) is totally justified for narrow resonances only.

A resonance formation is usually described by the Breit–Wigner cross section, see, e.g., [6]:

$$\sigma^{ik\to j}=\frac{\pi}{\mathbf{p}^{*2}}\frac{\Gamma^{ik\to j}\Gamma_j}{(\sqrt{S}-\bar{M}_j)^2+\Gamma_j^2/4},$$

where $\mathbf{p}^{*2}=F_{ik}^2/S$ is the CM momentum square, $\sqrt{S}$ is the CM energy, and Γ_j, $\Gamma^{j\to ik}$ are the total and partial decay widths, respectively. $\bar{M}_j$ is the average resonance mass. Substituting the above formula in (14), we find

$$\Delta^j(M_j^2)=\frac{1}{2\pi}\frac{\Gamma_j}{M_j}\frac{1}{(M_j-\bar{M}_j)^2+\Gamma_j^2/4}. \tag{19}$$

Thus, uniqueness of the profile function for the Breit–Wigner cross section has been demonstrated.

Considerations similar to those leading to formula (14) may be repeated for binary collisions. In this case the profile function is expressed through the cross sections of reaction $a+b \to \text{resonance} + c$ and the inverse one. Because the profile function arising from the binary collisions and the three-particle reactions (resonance formation and decay) has to be the same one we can get the relations between the cross sections of the different processes, where the resonance is involved.

IV. H-THEOREM AND AN EQUILIBRIUM STATE

The entropy production $\mathscr{H}$ is

$$\mathscr{H} = \partial^\mu S_\mu = \sum_k \int d^4\tilde{p}_k \ln f_k \cdot p_k^\mu \partial_\mu f_k.$$

Assuming that the distribution functions satisfy the kinetic equations (6), $p_k^\mu \partial_\mu f_k$ can be replaced by the collision terms of the right side of Eq. (6). If we assume that the transition rates for binary collisions and those of three-particle interactions satisfy the bilateral normalization conditions, we can consider separately the entropy production resulting from the binary collisions and the three-particle reactions. Anyhow it should be stressed that such an assumption is stronger than that arising from unitarity of the S-matrix, see the Appendix.

With the help of the bilateral normalization conditions one finds

$$\begin{aligned}\mathscr{H} = \mathscr{H}_{\mathrm{B}} + \sum_{i,j,k} \int \frac{d^3\mathbf{p}_i}{E_i} \frac{d^3\mathbf{p}_k}{E_k} d^4p_j\, \Delta^j(p_j^2) \\ \cdot \{[\kappa - \ln\kappa - 1]\, f_j W^{j\to ik}(p_j \,|\, p_i, p_k) \\ + [\kappa^{-1} + \ln\kappa - 1] f_i f_k W^{ik\to j}(p_i, p_k \,|\, p_j)\},\end{aligned}$$

where $\mathscr{H}_{\mathrm{B}}$ is the entropy production due to the binary reactions, see, e.g., [5]:

$$\kappa \equiv \frac{f_i(p_i)\, f_k(p_k)}{f_j(p_j)}.$$

The operations leading to the above formula are quite analogous to those described in Ref. [5]. It is seen that

$$\mathscr{H} \geqslant 0$$

and the entropy production vanishes when

$$f_i(p_i)\, f_k(p_k) = f_j(p_j) \qquad \text{for} \quad p_i + p_k = p_j. \tag{20}$$

Equilibrium, defined as a maximum entropy state, is reached when the distribution functions satisfy the functional relation (20). Standard considerations, see, e.g., Ref. [5], provide the Jüttner equilibrium function, i.e., a relativistic analogue of the Maxwell–Boltzmann distribution

$$f_j^{\text{eq}}(p) = \frac{g}{(2\pi)^3} \cdot \exp\left(\frac{\mu - u^\nu p_\nu}{T}\right), \tag{21}$$

where g is the number of internal degrees of freedom of a jth sort of particles, μ is the chemical potential, T is the temperature of the system, and u^ν is the four-velocity of the system as a whole. Thus, the form of the equilibrium distribution functions of stable and unstable particles is the same.

At the end of this section let us observe that the decay and formation processes provide an additional contribution to the entropy production. So, the presence of resonances in a system accelerates its equilibration and consequently makes the relaxation time shorter.

V. Macroscopic Characteristics of the Hadron Gas in Equilibrium

In this section we consider macroscopic characteristics like density and internal energy of the gas. We focus our attention on the resonance component of the gas. For simplicity we assume that particles do not carry any conserved charges. Thus, the numbers of particles are unlimited and the chemical potentials of all types of particles are equal to zero.

Using formulae (4) and (5), one finds the density and the internal energy density of a jth sort of resonance:

$$\begin{aligned} n_j &= \int d^4p\, \Delta^j(p^2)\, E f_j^{\text{eq}}(p), \\ e_j &= \int d^4p\, \Delta^j(p^2)\, E^2 f_j^{\text{eq}}(p). \end{aligned} \tag{22}$$

The equilibrium distribution function (21) in the rest frame of the system ($u^\nu = (1, 0, 0, 0)$) for $\mu = 0$ and $g = 1$ is

$$f_j^{\text{eq}}(p) = \frac{1}{(2\pi)^3}\, e^{-E/T}. \tag{23}$$

In formulae (22) we change the variables

$$(E, \mathbf{p}) \to (M, \mathbf{p}),$$

where $M^2 = E^2 - \mathbf{p}^2$. Putting (23) in (22) and integrating with respect to momenta, we get

$$\begin{aligned} n_j &= \int_0^{\infty} dMM\, \Delta^j(M^2) \left\{ \frac{1}{2\pi^2} TM^2 K_2\left(\frac{M}{T}\right) \right\}, \\ e_j &= \int_0^{\infty} dMM\, \Delta^j(M^2) \left\{ \frac{1}{2\pi^2} T^2 M^2 \left[\frac{M}{T} K_1\left(\frac{M}{T}\right) + 3K_2\left(\frac{M}{T}\right) \right] \right\}, \end{aligned} \tag{24}$$

where K_1 and K_2 are the so-called MacDonald functions [7]. In the parentheses under the integrals (24) one recognizes the density and internal energy density of stable particles, see, e.g., [5]. The resonance characteristics are those of stable particles averaged over the mass. One may wonder what is the normalization of the profile function. The explicit calculation shows that for the Breit–Wigner form (19)

$$\int dMM\, \Delta(M) = 1.$$

However, in this case the lower limit of the above integral has to be shifted to minus infinity. This operation is correct for the resonances with $\bar{M} \gg \Gamma$. Indeed the Breit–Wigner formula is of physical meaning for "narrow" resonances only.

Substituting the Breit–Wigner profile function (19) in formulae (24), we obtain

$$\begin{aligned} n_j &= \frac{1}{4\pi^3} T\Gamma_j \int_0^{\infty} dM \frac{M^2}{(M - \bar{M}_j)^2 + \Gamma_j^2/4} K_2\left(\frac{M}{T}\right), \\ e_j &= \frac{1}{4\pi^3} T^2 \Gamma_j \int_0^{\infty} dM \frac{M^2}{(M - \bar{M}_j)^2 + \Gamma_j^2/4} \left[3K_2\left(\frac{M}{T}\right) + \frac{M}{T} K_1\left(\frac{M}{T}\right) \right]. \end{aligned}$$

Since the above integrals cannot be calculated analytically, let us consider two limits.

For $\bar{M}_j \gg \Gamma_j$ and $T \gg \Gamma_j$ the functions $M^2 K_2$ and $M^2[3K_2 + (M/T) K_1]$, respectively, taken at $M = \bar{M}_j$, can be transferred from the integrals. Elementary integration provides the results

$$\begin{aligned} n_j &= \frac{1}{2\pi^2} T\bar{M}_j K_2(\bar{M}_j/T), \\ e_j &= \frac{1}{2\pi^2} T^2 \bar{M}_j [3K_2(\bar{M}_j/T) + (\bar{M}_j/T) K_1(\bar{M}_j/T)]. \end{aligned} \tag{25}$$

As would be expected, we have recovered the formulae for stable particles. It should be stressed that this result is not quite trivial as the procedure of determining the profile function is not trivial.

Instability should strongly manifest itself at $\Gamma_j \gg T$. Because we are interested in the qualitative effects of the mass smearing we use the Breit–Wigner formula for the "wide" resonance which is not quite correct. See the comment at the end of this section. We assume that Γ_j is of the order of $\bar{M}_j$ which additionally provides $\bar{M}_j \gg T$.

Under such conditions we can put the function $[(M-\bar{M}_j)^2+\Gamma_j^2/4]^{-1}$ taken at $M=0$ in front of the integrals. Then

$$n_j \cong \frac{3}{8\pi^2}\frac{T^4\Gamma_j}{\bar{M}_j^2+\Gamma_j^2/4}, \qquad e_j \cong \frac{3}{2\pi^2}\frac{T^5\Gamma_j}{\bar{M}_j^2+\Gamma_j^2/4}. \tag{26}$$

We have used the equality [7]

$$\int_0^\infty x^{\alpha-1}K_\nu(x)\,dx = 2^{\alpha-2}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right),$$

where $\Gamma(z)$ is the Euler gamma function and $\mathrm{Re}\,\alpha > |\mathrm{Re}\,\nu|$. Let us compare formulae (26) with analogous expressions for stable particles (formulae (25) for $\bar{M}_j \gg T$):

$$n_j^{\mathrm{st}} \cong \left(\frac{T\bar{M}_j}{2\pi}\right)^{3/2} e^{-\bar{M}_j/T}\left[1+\frac{15}{8}\frac{T}{\bar{M}_j}\right], \qquad e_j^{\mathrm{st}} \cong \left(\frac{T\bar{M}_j}{2\pi}\right)^{3/2} \bar{M}_j e^{-\bar{M}_j/T}\left[1+\frac{27}{8}\frac{T}{\bar{M}_j}\right]. \tag{27}$$

We see that the concentration of the resonances of average mass $\bar{M}$ highly exceeds the concentration of stable particles with mass $\bar{M}$.

It is a well-known experimental fact [2] that in hadron–hadron collisions at high energy there is an abundant resonance production as compared to pion yield. This abundance seems to decrease with incident energy. For many authors a big yield of relatively massive resonances was a crucial argument against thermodynamical approaches to particle production in hadron collisions since it was asserted that the generation of massive particles was exponentially suppressed according to formula (27). As shown, formula (27) can highly underestimate the resonance yield which seems to invalidate the above argumentation.

From formulae (26) one can find the energy per particle for a "wide" resonance at low temperature

$$\varepsilon \cong \tfrac{1}{4}T.$$

The above expression resembles the one for massless particles. It shows how important the effect of mass smearing can be.

At the end of this section a comment is in order. Our results concerning "wide" resonances are based on the Breit–Wigner profile function (19). There is a common consensus that the energy distribution of the resonance should be of the Breit-Wigner form near a maximum of the mass distribution. The problem of the distribution "tails," which are important for the validity of formulae (26), is cumbersome. There are rigorous arguments that the "tails" should deviate from the Breit–Wigner form while it is not clear how to modify them. For extensive discussion of the problem related to a non-exponential character of the decay law, see the review [8]. In the context of hadron resonances the problem of mass distribution has been discussed in Ref. [9].

We conclude this section as follows. While formulae (26) may be invalid due to uncertainties of the Breit–Wigner distribution "tails," the qualitative results of this section seem to be correct.

VI. Concluding Remarks

Let us discuss the assumptions leading to our kinetic theory model of hadron gas. The first important assumption occurs in the distribution function definition (3). Namely, we assume that the profile function is position-independent. As it has been argued in this way, quantum effects have been neglected. In the other case it would not be possible to determine the profile function with the help of formula (14). Since the profile function present in (3) is not specified, no other assumptions are made at this step of model formulation. Then, the kinetic equations have been considered and the collision terms have been defined. We have assumed that the profile function can be extracted from the transition rates in a way analogous to the extraction of the delta functions $\delta(p^2-m^2)$ for stable particles with mass m. The precise meaning of this operation is stated in formulae (8) and (9), where the transition rates are connected with the experimentally measurable quantities. Later on, no assumptions characteristic for our model are made.

The results of Section V are more or less obvious. Macroscopic characteristics of resonances are those of stable particles averaged over mass. Anyhow there are two important ingredients of formulae (24). It has been shown that the equilibrium functions of resonances coincide with those of stable particles. On the other hand, the profile function, i.e., the weight function in (24), has been uniquely determined.

We conclude as follows. The approach based on the distribution function definition (3) and the notion of profile function provides a self-consistent formalism very similar to the standard one and compatible with physical intuition concerning unstable particles.

Appendix

Unitarity of the S operator provides two equalities

$$\sum_{\beta} |\langle\beta| S |\alpha\rangle|^2 = \sum_{\alpha} |\langle\beta| S |\alpha\rangle|^2 = 1. \tag{A1}$$

From Eq. (A1) we get the bilateral normalization condition

$$\sum_{\alpha} (|\langle\beta| S |\alpha\rangle|^2 - |\langle\alpha| S |\beta\rangle|^2) = 0. \tag{A2}$$

Let us decompose the complete set of states α into states with definite number, N, of particles

$$\{\alpha\} = \sum_N \{\alpha_N\}.$$

We rewrite Eq. (A2) in the form

$$\sum_N \left(\sum_{\alpha_N} (|\langle\beta| S |\alpha_N\rangle|^2 - |\langle\alpha_N| S |\beta\rangle|^2) \right) = 0.$$

For determining the profile function and proving the H-theorem we have used the assumption that

$$\sum_{\alpha_N} (|\langle\beta| S |\alpha_N\rangle|^2 - |\langle\alpha_N| S |\beta\rangle|^2) = 0, \tag{A3}$$

which is stronger than Eq. (A2) arising from unitarity of the S-matrix. It is seen, however, that Eq. (A3) is strictly correct for interactions invariant under time inversion which is the case (at least approximately) for strong interactions.

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Note added in proof. The problems considered in this paper have been recently further studied (K. G. Denisenko and St. Mrówczyński, submitted to *Phys. Rev. C*). It has been shown that for equilibrium systems the concept of profile function follows from the S-matrix formulation of statistical mechanics by Dashen, Ma, and Berstein (*Phys. Rev.* **187** (1969), 345). Then, the equilibrium characteristics of the hadron gas of nucleons, pions, and deltas have been discussed. The finiteness of the delta decay width has been taken into account by means of realistic profile function.

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