



t -channel singularities in cosmology and particle physics

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Abstract

The t -channel singularity of a cross section for a binary $2 \rightarrow 2$ scattering occurs when the particle exchanged in the t -channel is kinematically allowed to be on its mass shell, that is when the process can be viewed as a sequence of two physical subprocesses: a two-body decay $1 \rightarrow 2$ and an inverse decay $2 \rightarrow 1$. We derive conditions for the singularity to be present in a binary process. A class of divergent cross sections for Standard Model processes has been determined and illustrated by the weak analog of the Compton scattering $Ze^- \rightarrow Ze^-$. After critically reviewing regularization methods proposed in literature, we discuss singular processes that occur in a medium composed of particles in thermal equilibrium. The medium is shown to regulate the singularities naturally as particles acquire a non-zero width. We demonstrate a possible cosmological application by calculating thermally averaged cross section for an elastic scattering within a simple scalar model. The transition probability, which is divergent in vacuum, becomes finite when the process occurs in a thermal bath due to the imaginary part of the t -channel mediator's self-energy computed within the Keldysh-Schwinger formalism.

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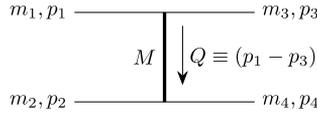


Fig. 1. The considered t -channel mediated process.

1. Introduction

Peierls noticed long ago [1] that when the hadron binary process $\pi N^* \rightarrow N^* \pi$ is described using the Feynman diagram with a proton exchange in the t -channel, the kinematics does not exclude the exchange of the proton that is on its mass shell. Then, the amplitude suffers from a divergence which also appears in a variety of binary reactions with unstable particles in the initial and final states. It was later realized [2] that the divergence occurs not only in binary but in any process if there is a phase-space point such that an amplitude of the processes can be viewed as a product of on-mass-shell amplitudes of real processes. In case of the reaction $\mu^+ \mu^- \rightarrow W^+ e^- \bar{\nu}_e$, which has received some attention [3–8] because of its importance for future $\mu^+ \mu^-$ colliders, the two possible subprocesses are $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$ and $\nu_\mu \mu^+ \rightarrow W^+$. The reaction $\mu^+ \mu^- \rightarrow W^+ e^- \bar{\nu}_e$, which proceeds via the muon-neutrino exchange in the t -channel, is thus kinematically possible with an on-mass-shell neutrino. Since the self-energy of a neutrino is purely real on its mass shell in vacuum, the resummed propagator does not regulate the divergence.

Even though a very existence of the singularities is an interesting phenomenon which is practically important in some specific cases, an awareness of the problem among high-energy physicists is rather limited. There are altogether only a few studies of the singularities (besides the publications mentioned above see [9,10]). In our opinion the existence of this yet unexplored divergence deserves a careful treatment and a satisfactory solution.

Here we discuss the problem of the t -channel divergences in detail. After reviewing two proposed regularization methods [3–8] which are, however, either not satisfactory or not universally applicable, we study the divergences in the context we have encountered them, i.e., when singular amplitude occurs not in vacuum but in an environment of gas particles. Specifically, working on a multi-component dark matter (DM) [11] we have found that for certain configurations of particle masses there exist singular transition probabilities which enter collision terms of Boltzmann equations.

We show here that presence of a medium naturally regularizes the t -channel singularities since mediating particles become quasi-particles, i.e., particles whose properties are modified by interaction with medium constituents. The modification includes a change of energy-momentum relation and an appearance of a non-zero width – the imaginary contribution to the self-energy that remains finite on the mass shell. To demonstrate how the regularization works we perform simple one-loop calculations within the Keldysh-Schwinger formalism, see, e.g., [12], showing that the thermally averaged transition probability is finite as long as the system’s temperature is non-zero.

2. t -channel singularity

At the beginning let us discuss in some detail when the singularity occurs in a binary process with the t -channel mediator of mass M . The process is depicted in Fig. 1. We are looking for

the condition in terms of masses and $s \equiv (p_1 + p_2)^2$ that has to be satisfied for the singularity to appear. One easily finds that $t = M^2$ corresponds to the fixed angle between \mathbf{p}_1 and \mathbf{p}_3 given as

$$\cos(\theta_{13}) = \frac{2E_1E_3 - m_1^2 - m_3^2 + M^2}{2\sqrt{E_1^2 - m_1^2}\sqrt{E_3^2 - m_3^2}}. \quad (1)$$

The requirement $|\cos(\theta_{13})| \leq 1$ implies the following necessary and sufficient condition

$$\Delta \equiv \beta^2 - 4\alpha\gamma \geq 0 \quad \text{and} \quad \sqrt{s_1} \leq \sqrt{s} \leq \sqrt{s_2}, \quad (2)$$

where $s_{1,2} \equiv (-\beta \mp \sqrt{\Delta})/(2\alpha)$, and

$$\begin{aligned} \alpha &\equiv M^2, & \beta &\equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2), \\ \gamma &\equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2m_4^2 - m_2^2m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2). \end{aligned}$$

It is also worth to specialize to the case of elastic scattering. When $m_1 = m_3$ and $m_2 = m_4$ we have $E_1 = E_3$ and $|\mathbf{p}_1| = |\mathbf{p}_3|$ in the center-of-mass frame, so $t \equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = 2m_1^2 - 2E_1^2 + 2|\mathbf{p}_1|^2 \cos(\theta_{13}) \leq 0$. Hence, no massive mediator ($M > 0$) can be on the mass shell. However, in the exchange channel, when $m_1 = m_4$ and $m_2 = m_3$, the singularity is possible. In a special case of the process $\nu_e W^+ \rightarrow W^+ \nu_e$ the condition (2) was derived in [10].

The condition (2) can be reformulated by saying that exactly one particle in the initial and in the final state must be unstable and the other must be stable. It can be proved that in such a case there exists a range of s that satisfies the condition (2). Here, the terms ‘unstable’ and ‘stable’ are restricted to the interactions shown in Fig. 1 only. If, in addition, the mediator is stable (in the standard sense), so that its width vanishes, the resummation of self-energies does not regularize the singularity.

3. Relevance of the singularity

There is a variety of Standard Model processes plagued by the divergence. We mention here the elastic scattering $W^- e^- \rightarrow e^- W^-$ mediated by the ν_e exchange as well as $Ze^- \rightarrow e^- Z$ and $W^+ \nu_e \rightarrow \nu_e W^+$, both with e being the mediator. The examples of inelastic processes are $Z\nu_e \rightarrow e^- W^+$ and $W^- \nu_e \rightarrow e^- Z$ mediated by e^+ and $\bar{\nu}_e$, respectively. Those processes can take place in collider scattering or during evolution of the early Universe.

Let us look closer at the $Ze^- \rightarrow e^- Z$ which is a weak analog of the Compton scattering with the photon being replaced by a Z boson. In this case, s_1 and s_2 , which enter the condition (2), are $s_1 \equiv 2m_Z^2 + m_e^2$ and $s_2 \equiv \frac{(m_Z^2 - m_e^2)^2}{m_e^2}$. In between s_1 and s_2 the cross section diverges. The length of the region is $s_2 - s_1 = m_Z^4/m_e^2(1 - 4m_e^2/m_Z^2) \gg m_Z^2$.

In Fig. 2 we plot the numerically computed total cross section $\sigma_{Ze \rightarrow eZ}$ as a function of \sqrt{s} . The contributing Feynman diagrams are depicted in the figure. The divergent region of \sqrt{s} that starts at $\sqrt{s} = \sqrt{s_1} \approx \sqrt{2}m_Z \approx 129$ GeV is clearly seen. The right edge of the divergent region located at $\sqrt{s} = \sqrt{s_2} \approx m_Z^2/m_e \approx 1.6 \cdot 10^7$ GeV is not shown.

Since the electron mass m_e is much smaller than the Z boson mass m_Z , the numerically found curve shown in Fig. 2 can be reproduced by the cross section of $Ze^- \rightarrow e^- Z$ computed analytically for $m_e = 0$. The analog of the Klein-Nishina formula [13] for the weak Compton scattering applicable for $m_Z^2 \lesssim s \lesssim 2m_Z^2$ reads:

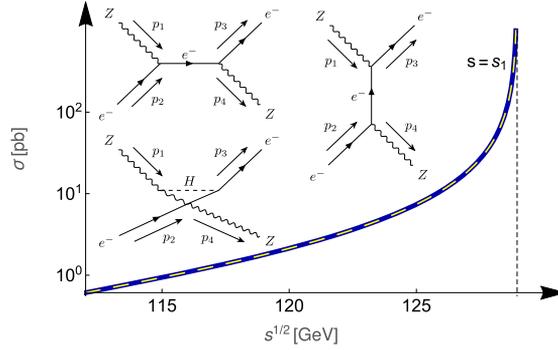


Fig. 2. The solid blue line shows the numerically integrated total cross section $\sigma_{Ze \rightarrow eZ}$ as a function of \sqrt{s} . The dashed yellow line represents the integrated Klein-Nishina-like formula for $m_e = 0$. The contributing Feynman diagrams are also depicted. The divergent region starts at $\sqrt{s} \approx \sqrt{2}m_Z \approx 129$ GeV, it ends at $\sqrt{s} = \sqrt{s_2} \approx m_Z^2/m_e \approx 1.6 \cdot 10^7$ GeV which is beyond the range of shown \sqrt{s} .

$$\frac{d\sigma_{Ze \rightarrow eZ}}{dt} = g^4 \frac{g_v^4 + 6g_a^2 g_v^2 + g_a^4}{12\pi c_w^4} \frac{(m_Z^4 - st)}{(s - m_Z^2)^2} \times \frac{m_Z^4 s^2 + 4m_Z^2(2m_Z^2 - s)st + (m_Z^4 - 4m_Z^2 s + s^2)t^2}{m_Z^4 s^2 t^2}, \tag{3}$$

where g is the $SU(2)$ gauge coupling constant, $c_w \equiv \cos \theta_W$, $g_v \equiv (1 - 4 \sin^2 \theta_W)/4$ and $g_a \equiv 1/4$. We note that in the limit $m_e \rightarrow 0$ the t -channel divergence appears at $t = 0$, $s_1 = 2m_Z^2$ and $s_2 = \infty$. However, it should be stressed that this is not the well known infrared divergence, as for $m_e \neq 0$ it emerges at non-zero t equal to m_e^2 . The total cross section shown by the dashed yellow line in Fig. 2, corresponds to $\sigma_{Ze \rightarrow eZ}(s) = \int_{2m_Z^2/s}^{m_Z^4/s} \frac{d\sigma}{dt} dt$.

As we also already mentioned, the analysis of the process $\pi N^* \rightarrow N^* \pi$ led to the discovery of t -channel singularities [1]. Numerous unstable hadron species provide many examples of singular hadronic processes whenever they could be modeled in terms of Feynman diagrams. The processes play an important role in relativistic heavy-ion collisions where numerous hadrons, which are produced, can be treated as a hadron gas [14]. Due to frequent hadron-hadron collisions the gas is in thermal equilibrium before the freeze-out. Since a proton is the only stable hadron, only processes with the proton exchange can suffer, strictly speaking, from the t -channel singularity. However, if a process with, say, π^+ exchange is regularized with its width controlled by weak interactions, the cross section becomes very big and the problem is still there.

In order to locate the t -channel singularity in a cosmological context we would like to mention the circumstances under which it has been noticed by us. In [11] we have proposed and discussed in detail a model of two-component dark matter. The model was an extension of the Standard Model by a spontaneously broken extra $U(1)_X$ gauge symmetry and a Dirac fermion. In order to generate mass for the dark vector X_μ the Higgs mechanism with a complex singlet was employed. Potential dark matter candidates were the massive vector boson X_μ and two Majorana fermions ψ_\pm , so the number of stable species was either 2 or 3. Adopting micrOMEGAS (version no 4.3.2; for the current version see [15]), which is a popular computational tool for dark matter studies, it has been noticed that for certain choices of dark particle masses solutions of Boltzmann equations for number densities of X_μ and ψ_\pm diverge. It turned out that the divergences appeared as a consequence of singularities of matrix elements described by t -channel Feynman

diagrams for scattering processes $\psi_+ X \rightarrow \psi_- h_{1,2}$, $\psi_+ \psi_- \rightarrow X h_{1,2}$ and $X \psi_{\pm} \rightarrow \psi_{\mp} h_{1,2}$ ($h_{1,2}$ denote two physical Higgs bosons), with an exchange of X , ψ_- and ψ_{\pm} in the t -channel, respectively. The singularities were present whenever the exchanged “virtual” particle had a chance to be on its mass shell. For parameters adopted in the publication [11] the singularities were not present.

Appearance of some singularities in a cosmological context has been also noticed in [16,17], where leptogenesis and freeze-in dark matter production were discussed, respectively. However, the authors focused mostly upon divergent s -channel amplitudes. Numerical difficulties implied by some t -channel diagrams have been recently mentioned by the authors of micrOMEGAs [15]. In the appendix of [15] there is discussed a differential cross section corresponding to a t -channel amplitude, but it seems that the range of parameters adopted ($\delta_{13} > 0$) precluded noticing the true t -channel singularity considered here.

From the review of existing literature, one can conclude that the issue of genuine (i.e., with non-zero mediator mass) t -channel singularities has not been discussed in a cosmological context. Sometimes, as, e.g., in [11], the singularity might have shown up, however specific range of parameters adopted made it absent. It should be emphasized that although in models of dark matter the presence of t -channel singularities is rather typical (since number of massive stable particles, which may work as mediators, increases), a systematic approach to the issue of t -channel singularities is, surprisingly, still missing.

4. Regularization by the mediator width

Let us now discuss how the issue of t -channel singularities has been addressed in context of scattering experiments. The first natural possibility is to regularize the divergence by a mediator width using the Breit-Wigner propagator with M^2 replaced by $M^2 - iM\Gamma$. Then, the t -channel amplitude is proportional to $(t - M^2 + iM\Gamma)^{-1}$, it is finite at $t = M^2$ and the cross section is proportional to $1/\Gamma$. The cross section is obviously infinite if the mediator is truly stable and $\Gamma = 0$. However, it is also abnormally big if a process of strong interactions is regularized by the width driven by weak interactions. Consequently, the cross section of a hadronic process with, e.g., π^+ exchange becomes orders of magnitude bigger than the typical hadronic cross section at $t = M^2$. So, it is a doubtful regularization technique.

5. Stable mediator – known approaches

Ginzburg [3] proposed another method which originates from the observation that some initial-state particles are unstable. Assuming that the particle ‘1’ decays as $1 \rightarrow \Phi + 3$ with the width Γ_1 (here, Φ denotes the t -channel mediator), the particle wave function in the rest frame should include, according to the method [3], the factor

$$e^{-im_1 t} e^{-\Gamma_1 t} = e^{-i(m_1 - i\Gamma_1)t} . \quad (4)$$

Hence, we can treat $\tilde{m}_1 \equiv m_1 - i\Gamma_1$ as a complex mass of the particle ‘1’. After a Lorentz boost, its four-momentum is also complex

$$(\tilde{E}_1, \tilde{\mathbf{p}}_1) = \left(1 - i \frac{\Gamma_1}{m_1}\right) (E_1, \mathbf{p}_1) . \quad (5)$$

Consequently, the variable t is complex as well

$$\begin{aligned}
 t &= (\tilde{E}_1 - E_3)^2 - (\tilde{\mathbf{p}}_1 - \mathbf{p}_3)^2 \\
 &= \left(E_1 - E_3 - i \frac{\Gamma_1}{m_1} E_1 \right)^2 - \left(\mathbf{p}_1 - \mathbf{p}_3 - i \frac{\Gamma_1}{m_1} \mathbf{p}_1 \right)^2,
 \end{aligned}
 \tag{6}$$

and the singularity condition $t = M^2$ is never satisfied. Unfortunately, the Ginsburg’s method contradicts the exact energy-momentum conservation. Indeed, the real part of t is

$$\Re t = (E_1 - E_3)^2 - (\mathbf{p}_1 - \mathbf{p}_3)^2 - \Gamma_1^2,
 \tag{7}$$

but using $p_1 - p_3 = p_4 - p_2$ and observing that the particle ‘4’ is also unstable with the width Γ_4 , one gets the expression (7) with Γ_1 replaced by Γ_4 . Since, in general, $\Gamma_1 \neq \Gamma_4$, there appears an energy-momentum mismatch. Therefore, the complex mass solution is rather unsatisfactory.

An alternative method suggested by Melnikov and Serbo [4] and further developed in [5,6], see also [7,8], makes use of a finite size of a beam of long-living particles. The method was worked out in the context of a future $\mu^+ \mu^-$ collider. The cross section is then proportional not to the modulus squared of matrix element $|\mathcal{M}_{fi}|^2$ but to $\mathcal{M}_{fi}^* \mathcal{M}_{fi'}$ where the final state is the same but initial momenta slightly differ. The difference is of the order of $1/a$ with a being the transverse size of the initial beam. Consequently, the singular points of the amplitude \mathcal{M}_{fi} and $\mathcal{M}_{fi'}$ are slightly shifted from each other and the divergent integral is changed into

$$\int \frac{dt}{|t - M^2 + i\varepsilon|^2} \rightarrow \int \frac{dt}{(t - \Pi - M^2 + i\varepsilon)(t + \Pi - M^2 - i\varepsilon)},
 \tag{8}$$

where the integration is performed over a vicinity of the point $t = M^2$ and $\Pi = m/a$ with m being the mass of incoming particles. For a symmetric Gaussian beam profile $\sqrt{2/\pi} a$ is equal to the Gaussian width of the beam. One shows that the regularized integral is proportional to $1/\Pi = a/m$.

The beam-size method has a solid foundation and is applicable to a collider experimental situation. However, it is not suitable if the singular process occurs in a particle gas as in the cosmological context. In this case transition probabilities that enter Boltzmann equations describe scatterings of statistically distributed particles moving at various directions and energy-momenta.

6. Thermal-field-theory regularization

Particles propagating and interacting with gas constituents become quasi-particles because of their finite mean free path or mean free flight time. Particle self-energies, which are purely real on the mass shell in vacuum, acquire imaginary contributions in medium. Consequently, quasi-particles have non-zero widths which allow one to regularize the amplitudes that suffer from t -channel singularities. We emphasize that in a medium there is no fundamental difference between particles that are unstable in vacuum and those that are stable, as all particles are actually quasi-particles of non-zero width.

Before we will explicitly show how the t -channel divergences are regularized in the thermal field theory, let us briefly discuss the problem in a broader context of divergences that plague the thermal field theory. Ultraviolet (UV) divergences are as in vacuum but infrared (IR) divergences are often much more severe because a distribution function of massless bosons, which enters a variety of quantities like transition rates, diverges as $1/E$ where E is the energy of the massless boson. However, there has been developed a systematic procedure to tame the IR divergences, known as the Hard Thermal Loop approach [18] which is usually applied as follows. An IR divergent quantity is split into the soft and hard parts. The divergent soft part is computed with

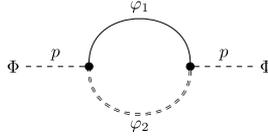


Fig. 3. The one-loop diagram for the self energy of Φ .

the resummed propagator of a massless mediator. If the mediator momentum is space-like the divergence is regulated by the thermal or Debye mass. In case of the time-like momentum of the mediator the imaginary part of its self-energy is needed to remove the divergence. Subsequently one combines the soft and hard parts and the final result should be independent of the momentum parameter which was used to separate the soft part from the hard one. The procedure, which sometimes requires using not only the resummed propagators but the resummed vertices as well, has been repeatedly used in astroparticle physics. As examples, we mention the studies on production of gravitons [19,20], gravitinos [21], axions [22,23] and right-handed neutrinos [24–26].

It should be stressed that the t -channel singularity we are interested in is neither IR nor UV, as long the mediator mass M is non-zero. The singularity occurs at $t = M^2$. However, when characteristic momenta of the process under consideration are much bigger than M and the mediator is treated as massless the t -channel singularity is shifted into the IR domain. Our treatment of the process $Ze^- \rightarrow Ze^-$ and the cross section shown in Fig. 2 illustrate such a situation. Then, the t -channel singularity can be regulated using the HTL procedure and it does not matter what the actual origin of the divergence is. In our opinion this is what happens in the studies [24–26]. The t -channel singularities are moved into the IR domain and therefore their relevance as a new kind of divergence has not been discussed.

We are going to demonstrate how to regularize the t -channel divergence which appears when $t \rightarrow M^2$ and M is non-zero. We use the Keldysh-Schwinger formalism, see, e.g., [12], applicable both to equilibrium and non-equilibrium systems of quantum fields. A toy model that we adopt for illustrative calculations consists of three real scalar fields: $\varphi_{1,2}$ and Φ . The Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \left[(\partial^\mu \varphi_1)(\partial_\mu \varphi_1) - m_1^2 \varphi_1^2 \right] + \frac{1}{2} \left[(\partial^\mu \varphi_2)(\partial_\mu \varphi_2) - m_2^2 \varphi_2^2 \right] + \frac{1}{2} \left[(\partial^\mu \Phi)(\partial_\mu \Phi) - M^2 \Phi^2 \right] + \mu \varphi_1 \varphi_2 \Phi, \tag{9}$$

where masses of φ_1 , φ_2 and Φ are denoted as m_1 , m_2 and M , respectively, and μ is an interaction coupling of mass dimension. We assume that $m_1 > m_2 + M$, so that φ_2 and Φ are stable (in cosmological applications they would be candidates for DM particles) while φ_1 decays as $\varphi_1 \rightarrow \varphi_2 \Phi$. Consequently, the amplitude of the process $\varphi_1 \varphi_2 \rightarrow \varphi_2 \varphi_1$ mediated by Φ in the t -channel is singular. Since the mass M is kept finite and non-zero the divergence is not hidden in the IR domain.

We are going to compute the one-loop self-energy of the field Φ represented by the graph shown in Fig. 3. Although our reasoning does not require the assumption of thermal equilibrium since the Keldysh-Schwinger formalism works both for equilibrium and non-equilibrium systems, we consider the quantum fields φ_1 , φ_2 and Φ in global thermodynamic equilibrium to simplify the problem. Hence, we assume that a quasi-particle Φ propagates across the thermal bath of temperature T interacting with the system’s constituents.

We are interested in a nonzero imaginary contribution to Φ self-energy on the mass shell, as it is needed to regularize the singular transition probability. That contribution usually corresponds

to a physical scattering process and it appears at the two-loop level when binary interactions contribute to the self-energy. However, here the contribution of interest is present already at the one-loop level due to the decay $\varphi_1 \rightarrow \varphi_2 \Phi$.

For the t -channel singularity to appear, particles unstable in vacuum must be present both in the initial and in the final states. Therefore, an intuition suggests that a solution of the problem should be related to the very same instability, that, on the other hand, implies a lack of proper asymptotic in and out states in vacuum. It will be shown below that indeed the treatment proposed here is intrinsically related to the instability. It is worth mentioning that in general, in statistical quantum field theory, there are no asymptotic states regardless if they are made of states that are stable or unstable in vacuum since in a remote past and in a remote future all particles do interact with their environment.

As explained in detail in, e.g., [27], the one-loop retarded self-energy $\Pi^+(p, T)$ we are interested in is

$$\Pi^+(p, T) = \frac{i\mu^2}{2} \int \frac{d^4k}{(2\pi)^4} \left[\Delta_1^+(k+p) \Delta_2^{\text{sym}}(k, T) + \Delta_1^{\text{sym}}(k, T) \Delta_2^-(k-p) \right], \quad (10)$$

where the indices 1 and 2 refer to the fields φ_1 and φ_2 , respectively. The free Green's functions $\Delta^\pm(p)$ and $\Delta^{\text{sym}}(p, T)$ of a scalar field of mass m are

$$\Delta^\pm(p) = \frac{1}{p^2 - m^2 \pm i \operatorname{sgn}(p_0) \varepsilon}, \quad (11a)$$

$$\Delta^{\text{sym}}(p, T) = -\frac{i\pi}{E_p} \left(\delta(E_p - p_0) + \delta(E_p + p_0) \right) [2f(E_p, T) + 1], \quad (11b)$$

where $E_p \equiv \sqrt{\mathbf{p}^2 + m^2}$ and $f(E_p, T)$ is the Bose-Einstein distribution function $f(E_p, T) = (e^{\beta E_p} - 1)^{-1}$ with $\beta \equiv 1/T$.

Our calculations are performed in the rest frame of the heat bath which in cosmology corresponds to the reference frame where the Universe is homogeneous and isotropic. However, after replacing βE_p by $\beta u^\mu p_\mu$ our calculations would be valid in a reference frame which moves with respect to the heat bath with the four-velocity u^μ .

Substituting the functions (11a) and (11b) into Eq. (10), the imaginary part $\Sigma(\mathbf{p}, T) \equiv \Im \Pi^+(p, T)$ for the on-mass-shell momentum, i.e., for $p_0 = E_p \equiv \sqrt{\mathbf{p}^2 + M^2}$, is obtained as

$$\begin{aligned} \Sigma(\mathbf{p}, T) = & -\frac{\pi}{4} \mu^2 \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{e^{\beta E_{k2}} + 1}{e^{\beta E_{k2}} - 1} \right. \\ & \times \left[\frac{\operatorname{sgn}(E_{k2} + E_p)}{E_{k2}} \delta(M^2 + m_2^2 - m_1^2 + 2E_{k2}E_p - 2\mathbf{k} \cdot \mathbf{p}) \right. \\ & \left. \left. + \frac{\operatorname{sgn}(-E_{k2} + E_p)}{E_{k2}} \delta(M^2 + m_2^2 - m_1^2 - 2E_{k2}E_p - 2\mathbf{k} \cdot \mathbf{p}) \right] \right. \\ & \left. + [1 \leftrightarrow 2, \mathbf{k} \rightarrow -\mathbf{k}] \right\}, \quad (12) \end{aligned}$$

where $E_{k1} \equiv \sqrt{\mathbf{k}^2 + m_1^2}$ and $E_{k2} \equiv \sqrt{\mathbf{k}^2 + m_2^2}$. After tedious calculations the following result has been found

$$\Sigma(|\mathbf{p}|, T) = -\frac{\mu^2}{16\pi} \frac{1}{\beta|\mathbf{p}|} \times \left[\ln \frac{e^{\beta(A+C)} - 1}{e^{\beta A} - 1} - \ln \frac{e^{\beta(A+B+C)} - 1}{e^{\beta(A+B)} - 1} \right], \quad (13)$$

where

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - 2Mk_*|\mathbf{p}|}{2M^2},$$

$$B \equiv E_p, \quad C \equiv \frac{2k_*|\mathbf{p}|}{M}, \quad k_* \equiv \frac{\sqrt{\lambda(m_1^2, m_2^2, M^2)}}{2M},$$

and $\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. Since the system is isotropic the self-energy depends only on $|\mathbf{p}|$ not \mathbf{p} . Note that the imaginary part of the self-energy (13) vanishes in the zero-temperature limit.

In Fig. 4 we present the thermally averaged cross section for $\varphi_1 \varphi_2 \rightarrow \varphi_2 \varphi_1$ process mediated by Φ in the t -channel. This quantity is defined as

$$(\sigma v)(T) = \mu^4 \int d\Pi_1 d\Pi_2 \frac{f_1(E_1, T)}{n_1(T)} \frac{f_2(E_2, T)}{n_2(T)} \times \int d\Pi_3 d\Pi_4 \frac{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)}{(t - M^2)^2 + \Sigma^2(|\mathbf{p}_{13}|, T)}, \quad (15)$$

where $\mathbf{p}_{13} \equiv \mathbf{p}_1 - \mathbf{p}_3$, while $f_k(E_k, T)$ and $n_k(T)$ are, respectively, the Bose-Einstein distribution function and the equilibrium number density of particle species φ_k ($k = 1, 2$), and $d\Pi_k \equiv d^3 p_k / [(2\pi)^3 2E_k]$ denotes a phase-space element for $k = 1, 2, 3, 4$. The parameters are chosen as $m_1 = 70$ GeV, $m_2 = 40$ GeV, $M = 20$ GeV, and $\mu = 10$ GeV which are of the order of magnitude relevant for the dark matter studies [11]. Strictly speaking, the thermally averaged cross section is not a cross section as it does not describe transitions between asymptotic states. However, it approximates the cross section in the limit of a dilute system.

Eq. (15) clearly shows that the thermally averaged cross section is divergent if the equality $t = M^2$ is kinematically allowed and $\Sigma^2(|\mathbf{p}_{13}|, T)$ is absent. The integration variables can be changed in order to explicitly integrate over s , with the lower integration limit $s_{\min} \equiv \max[(m_1 + m_2)^2, (m_3 + m_4)^2]$. Therefore, whenever $s_{\min} < s_2$ and $\Delta > 0$ (see condition (2)), the integration hits the divergent region corresponding to $t = M^2$, so that the integrand diverges in some range of s . The non-zero imaginary part of self-energy removes the divergence. We note that the one-loop self energy includes not only the imaginary but also the real part which shifts the mass M . However, the shift is irrelevant for our considerations so the real part of self-energy is neglected.

As seen in Fig. 4 the thermally averaged cross section quickly increases with $x \equiv M/T$. It is a consequence of the thermal width $\Gamma(|\mathbf{p}_{13}|, T) \equiv \Sigma(|\mathbf{p}_{13}|, T)/M$ diminishing with decreasing temperature. In the limit of zero temperature the thermal width vanishes and the cross section blows up. For comparison we have also shown the cross section calculated for a few selected constant widths. It is worth noticing that for small x , i.e., $x \lesssim 0.1$, $\Gamma(|\mathbf{p}_{13}|, T)$ could be approximated by a constant width $\Gamma(|\mathbf{p}_{13}|, T) \approx 0.1$ GeV using the Breit-Wigner formula.

7. Summary and conclusions

The t -channel singularities have been discussed paying particular attention to binary processes. We have formulated conditions for the singularity to appear and the singular Standard

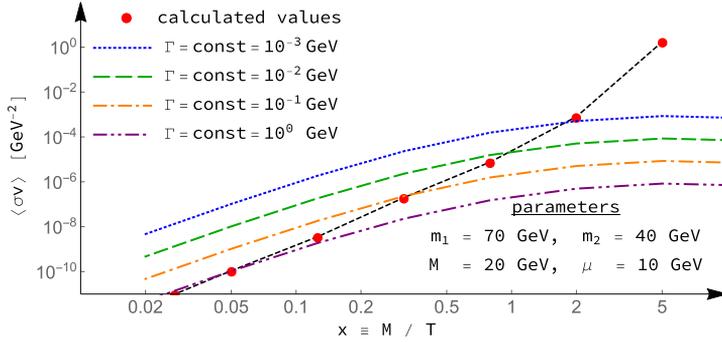


Fig. 4. The thermally averaged cross section for the process $\varphi_1 \varphi_2 \rightarrow \varphi_2 \varphi_1$. The red points show results of Monte Carlo phase-space integration in (15) and the dashed black line is a polynomial fit drawn to guide reader's eyes. The dotted blue, dashed green, dashed-dotted orange and dashed-double-dotted purple lines show results obtained adopting the Breit-Wigner formula with a constant width $\Gamma(|\mathbf{p}_{13}|, T) = 10^{-3}, 10^{-2}, 10^{-1}, 10^0$ GeV, respectively.

Model process $Ze^- \rightarrow e^- Z$ has been discussed in detail. After reviewing critically regularization methods proposed in literature we have focused on singular processes which occur in a gas of particles. It has been argued that the medium naturally regulates the singularity as particles become quasi-particles with a non-zero width.

The singularity is neither IR nor UV, as long the mediator mass M is finite and non-zero. However, if the mass is neglected the t -channel singularity is shifted into the IR domain and it can be regulated using the HTL technique as the actual origin of the divergence does not matter. A specific character of the singularity becomes evident when $M \neq 0$ and, therefore, we have focused on this case. To illustrate the regularization procedure we have computed, using the Keldysh-Schwinger formalism, the imaginary part of the self-energy of a scalar mediator and a thermally averaged cross section for the process that is singular in vacuum. It has been shown that the cross section becomes finite when the process occurs in the thermal bath.

The singularity is present in the Standard Model and its extensions. In particular, it appears naturally in multicomponent dark matter scenarios (e.g., in the vector-fermion model discussed in [11]), as they introduce massive stable particles that can be produced in decays of other species.

It should be emphasized that even when the singularity is regularized by the thermal width, the cross section is still large. This means that the discussed processes cannot be neglected (or even: can play a crucial role) during, e.g., calculation of cosmological relic densities of the particles involved. Hence, correctness of the regularization procedure is potentially of great importance.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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