CHROMOHYDRODYNAMICS OF THE QUARK PLASMA

Stanisław MRÓWCZYŃSKI [1]
Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20C, SF-00170 Helsinki, Finland

Received 8 December 1987

The equations of ideal hydrodynamics of the quark plasma interacting with the chromodynamic mean field are derived from kinetic theory. It is shown that the non-abelian effects disappear from the equations.

Several versions of the hydrodynamical model have been extensively used to study the evolution of a quark–gluon plasma produced (if indeed produced) in ultrarelativistic heavy-ion collisions (for a review see ref. [1]). In these studies the plasma has been assumed locally colorless and the chromodynamic forces have not influenced the liquid motion. On the other hand it is commonly believed that at the initial stages of hadron collisions there is color separation (in space), and the chromodynamic fields play a nontrivial role leading, in particular, to parton-pair generation from vacuum. The latter effects have been considered in the Low–Nussinov [2] or Lund [3] type models of hadron collisions, which have been recently adopted [4,5] to nuclear interactions at very high energies.

The chromohydrodynamics joins both approaches quoted above since it describes the evolution of the colored quark–gluon liquid interacting with the self-consistently generated chromodynamic field. Very recently the chromohydrodynamic model has been applied to study the quark–gluon plasma [5]; however, the equations of the electrodynamic plasma have been used there.

In this letter we derive the equations of ideal hydrodynamics of a quark plasma i.e. the system of colored quarks interacting via classical non-abelian potentials. Our starting point is the gauge covariant kinetic theory which has been recently formulated [6]. The derivation of the chromohydrodynamic equations is given in ref. [7]. However, the approach is essentially gauge non-covariant since the hamiltonian formalism is used and consequently the (temporal axial) gauge is fixed from the beginning. Furthermore, the authors of ref. [7] have exploited the concept of the classical continuous color variable, which is, in our opinion, rather unfortunate since there is no analog of this variable in the microscopic field theory, i.e. QCD.

Because the kinetic theory [6] forms the basis of our considerations, let us start with a brief presentation of it. For simplicity we consider spinless quarks of one flavour only. The (anti-)quark distribution function $f(p,x)$ ($\bar{f}(p,x)$) is a 3×3 matrix in color space and transforms under local gauge transformations as

$$f(p,x)\mapsto U(x)\,f(p,x)\,U^{+}(x)\ . \tag{1}$$

The trace of the distribution function is, of course, gauge invariant. f and $\bar{f}$ satisfy the transport equations [the units are used where $c=k_{\rm B}=\hbar=1$; the signature of the metric tensor is $(+,-,-,-)$]

$$p^{\mu}\mathrm{D}_{\mu}f(p,x)-\tfrac{1}{2}gp^{\mu}(\partial/\partial p_{\nu})\{F_{\mu\nu}(x),f(p,x)\}=C[f,\bar{f}]\ , \tag{2a}$$

$$p^{\mu}\mathrm{D}_{\mu}\bar{f}(p,x)+\tfrac{1}{2}gp^{\mu}(\partial/\partial p_{\nu})\{F_{\mu\nu}(x),\bar{f}(p,x)\}=\bar{C}[f,\bar{f}]\ , \tag{2b}$$

where $p^{\mu}\equiv p=(E,\boldsymbol{p})$; D^{μ} is the covariant derivate in adjoint representation which acts as $\partial^{\mu}+ig[A^{\mu}(x),\ ]$;

[1] Permanent address: High Energy Department, Institute for Nuclear Studies, Hoza 69, PL-00-681 Warsaw, Poland

0370-2693/88/$ 03.50 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

$\{\,,\}$ denotes the anticommutator and $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+\mathrm{i}g[A^\mu,A^\nu]$ is the stress tensor of the chromodynamic field generated by the color-quark current

$$D_\mu F^{\mu\nu}(x)=j^\nu(x)\,, \tag{3}$$

where

$$j^\mu(x)=\frac{g}{2}\int\frac{d^3p}{(2\pi)^3E}p^\mu\{f(p,x)-\bar{f}(p,x)$$
$$-\tfrac{1}{3}\mathrm{Tr}[f(p,x)-\bar{f}(p,x)]\}\,. \tag{4}$$

(If matrix and scalar quantities are added to one another it is understood that the scalar quantity is multiplied by the unit matrix.) C and $\bar{C}$ are the collision terms, certain gauge-covariant forms of which have been discussed in our very recent papers [8].

In fact, eqs. (2) form only the classical limit of the exact quantum equations derived by Elze, Gyulassy and Vasak [6]. Therefore, we consequently treat quarks as classical particles with Boltzmann statistics. However, as briefly discussed at the end of the paper, some effects of Fermi statistics of quarks can be easily taken into account in our considerations.

The ideal hydrodynamics, which is discussed in this paper, corresponds to the plasma in local equilibrium, while the local equilibrium is determined by the maximum of the entropy density. We define the gauge-invariant entropy four-flow as

$$S^\mu(x)=\int\frac{d^3p}{(2\pi)^3E}p^\mu\{\mathrm{Tr}\,f(p,x)[\ln\mathrm{Tr}\,f(p,x)-1]$$
$$+\mathrm{Tr}\,\bar{f}(p,x)[\ln\mathrm{Tr}\,\bar{f}(p,x)-1]\}\,.$$

Then, using eqs. (2) and assuming that the distribution functions vanish at infinite momenta, one easily finds that $\partial_\mu S^\mu=0$ if $\mathrm{Tr}\,C=\mathrm{Tr}\,\bar{C}=0$. Here we do not specify C and $\bar{C}$ since it is enough for our purposes only to demand that $\mathrm{Tr}\,C=0$ and $\mathrm{Tr}\,\bar{C}=0$ if

$$f(p_1,x)\,f(p_2,x)=f(p_3,x)\,f(p_4,x)\,, \tag{5a}$$

$$f(p_1,x)\,\bar{f}(p_2,x)=f(p_3,x)\,\bar{f}(p_4,x)\,, \tag{5b}$$

$$\bar{f}(p_1,x)\,\bar{f}(p_2,x)=\bar{f}(p_3,x)\,\bar{f}(p_4,x)\,, \tag{5c}$$

where $p_1+p_2=p_3+p_4$.

Using standard arguments [9], one finds the local equilibrium distribution functions

$$f^{\mathrm{eq}}(p,x)=\rho(x)\exp[-u^\mu(x)\,p_\mu/T(x)]\,, \tag{6}$$

$$\bar{f}^{\mathrm{eq}}(p,x)=\bar{\rho}(x)\exp[-u^\mu(x)\,p_\mu/T(x)]\,, \tag{6 cont'd}$$

where $u^\mu(x)$ is the hydrodynamic velocity and $T(x)$ is the temperature measured in the local rest frame. Because $f(p,x)$ and $\bar{f}(p,x)$ are matrices in color space, so are $\rho(x)$, $\bar{\rho}(x)$ and $u^\mu(x)$. However, all these matrices have to commute with one another to satisfy eqs. (5).

Because the distribution functions are gauge dependent, the same holds for the matrices $\rho(x)$, $\bar{\rho}(x)$ and $u^\mu(x)$. Assuming that these matrices transform under gauge transformation as tensors, i.e. according to (1), one finds that the local equilibrium distribution functions transform as is required by eq. (1).

Additionally we assume that

$$\int\frac{d^3p}{(2\pi)^3E}X(p)\,C[f,\bar{f}]=0\,,$$
$$\int\frac{d^3p}{(2\pi)^3E}\bar{X}(p)\,\bar{C}[f,\bar{f}]=0\,, \tag{7}$$

where $X(p)$ denotes a quantity conserved in binary collisions of quarks and antiquarks.

Integrating eqs. (2) over $d^3p/(2\pi)^3E$ one gets

$$D_\mu N^\mu_+(x)=0\,,\quad D_\mu N^\mu_-(x)=0\,, \tag{8}$$

where

$$N^\mu_+(x)=\int\frac{d^3p}{(2\pi)^3E}p^\mu f(p,x)\,,$$
$$N^\mu_-(x)=\int\frac{d^3p}{(2\pi)^3E}p^\mu\bar{f}(p,x)\,. \tag{9}$$

We have used eqs. (7) with $X=\bar{X}=1$ which corresponds to the particle number conservation in a binary collision. It has also been assumed that the distribution functions vanish at infinite four-momenta. Taking the trace of eqs. (8) one finds the baryon current conservation equation

$$\partial_\mu j^\mu_b(x)=0\,, \tag{10}$$

with

$$j^\mu_b(x)=\tfrac{1}{3}\mathrm{Tr}\,N^\mu(x)\,,$$

where

$$N^\mu(x)=N^\mu_+(x)-N^\mu_-(x)\,.$$

Combining eqs. (8) and (10) one finds the covariant conservation of the quark color current (4), namely

$$D_\mu j^\mu(x)=0\,,$$

since

$$j^\mu(x)=\tfrac{1}{2}g[N^\mu(x)-\tfrac{1}{3}\mathrm{Tr}\,N^\mu(x)]\,. \tag{11}$$

Adding the left-hand and the right-hand sides of eqs. (2a) and (2b), multiplying the resulting equation by p and performing the integration over $d^3p/(2\pi)^3E$, one gets the second-moment equation

$$D^\mu\Theta_{\mu\nu}(x)+\tfrac{1}{2}g\{F_{\mu\nu}(x),N^\mu(x)\}=0\,, \tag{12}$$

where

$$\Theta^{\mu\nu}(x)=\int\frac{d^3p}{(2\pi)^3E}p^\mu p^\nu[f(p,x)+\bar{f}(p,x)]\,. \tag{13}$$

Eqs. (8) and (13) constitute the ideal hydrodynamic equations if the moments are calculated with the local equilibrium distribution functions given in eq. (6).

Repeating simple considerations from ref. [9] one gets

$$N^\mu_+(x)=n_+(x)\,u^\mu(x)\,,$$

$$N^\mu_-(x)=n_-(x)\,u^\mu(x)\,, \tag{14}$$

where $n_+(x)$ and $n_-(x)$ are the (matrix) densities of quarks and antiquarks, respectively. Then,

$$\Theta^{\mu\nu}(x)=\{e(x)[n_+(x)+n_-(x)]+p(x)\}u^\mu(x)\,u^\nu(x)$$
$$-\,p(x)\,g^{\mu\nu}\,, \tag{15}$$

where $e(x)$ is the (scalar) energy density per particle and

$$e(x)=m\frac{K_1(m/T(x))}{K_2(m/T(x))}+3T(x)\,, \tag{16}$$

m is the quark mass and K_i are the so-called MacDonald functions. For massless quarks $e(x)=3T(x)$; $p(x)$ is the matrix pressure related to the quark density by the ideal gas equation of state

$$p(x)=[n_+(x)+n_-(x)]\,T(x)\,. \tag{17}$$

It can be surprising that the pressure is a gauge-dependent matrix. However, one should note that the mechanical pressure $P(x)=\mathrm{Tr}\,p(x)$, and it is gauge invariant. Therefore, as in the case of the mixture of ideal gases, the pressure is a sum of terms related to the mixture components.

Eqs. (8), (12) and (3) with eqs. (14)–(17) and (11) form the gauge-covariant set of hydrodynamic equations of an ideal quark plasma. To make the set complete one has to add the equation expressing the isotropic character of an ideal fluid motion. The set can be essentially simplified by the proper choice of a gauge.

As quoted previously, the matrices $\rho(x)$, $\bar{\rho}(x)$, and $u^\mu(x)$, which are hermitian, transform under unitary gauge transformations according to eq. (1). Therefore they can be diagonalized simultaneously (because they commute with one another) by means of a gauge transformation. This is just our gauge condition. Further, one finds that having the diagonal $N^\mu_+(x)$ and $N^\mu_-(x)$, eqs. (8) are decomposed into the differential equations, where enter the diagonal components of $A^\mu(x)$, and into the algebraic equations with the off-diagonal components of the four-potential. Then, it follows from these algebraic equations that the off-diagonal components of $A^\mu(x)$ have to vanish. Therefore, except $n(x)$ and $u^\mu(x)$, $A^\mu(x)$ is also diagonal and so is the stress tensor.

If we introduce the indices i, j which run over the diagonal components (1,2,3) of all quantities of interest, the hydrodynamic equations read (summation over repeated i, j indices is not implied here)

$$\partial_\mu N^\mu_{i+}(x)=0\,,\quad \partial_\mu N^\mu_{i-}(x)=0\,, \tag{18}$$

$$\partial_\mu\Theta^{\mu\nu}_i(x)+gF^{\mu\nu}_i(x)\,N_{\mu i}(x)=0\,, \tag{19}$$

$$\partial_\mu F^{\mu\nu}_i(x)$$
$$=\tfrac{1}{2}g\{N^\nu_i(x)-\tfrac{1}{3}[N^\nu_1(x)+N^\nu_2(x)+N^\nu_3(x)]\}, \tag{20}$$

where

$$N^\mu_{i+}(x)=n_{i+}(x)u^\mu_i(x)\,,$$

$$N^\mu_{i-}(x)=n_{i-}(x)u^\mu_i(x)\,, \tag{21}$$

$$N^\mu_i(x)=N^\mu_{i+}(x)-N^\mu_{i-}(x)\,, \tag{22}$$

$$\Theta^{\mu\nu}_i(x)$$
$$=\{e(x)[n_{i+}(x)+n_{i-}(x)]+p_i(x)\}u^\mu_i(x)u^\nu_i(x)$$
$$-p_i(x)\,g^{\mu\nu}\,, \tag{23}$$

and

$$p_i(x)=[n_{i+}(x)+n_{i-}(x)]\,T(x)\,. \tag{24}$$

Eq. (16) remains unchanged. Because the stress tensor and color current are traceless, only two among three equations (20) are independent from each other. Therefore, eqs. (20) connect two components ($F_3^{\mu\nu}$ and $F_8^{\mu\nu}$ in adjoint representation) of $F^{\mu\nu}$ with three components of N^{μ}. Eqs. (18)–(24) constitute the non-covariant version of the chromohydrodynamic equations.

In our considerations quarks have been treated as classical particles admitting Boltzmann statistics. If one uses the Fermi–Dirac distribution functions of local equilibrium instead of the Maxwell–Boltzmann ones (6), then the set of hydrodynamic equations remains unchanged, except eqs. (16) and (17), which are modified in a well-known way.

In the paper by Gatoff, Kerman and Matsui [5] the chromohydrodynamic evolution of the quark–gluon plasma produced in ultrarelativistic heavy-ion collisions has been studied in the flux-tube model, which assumes that the initial nuclei are color charged at the instant of collision due to the exchange of some soft gluons. In the light of the discussion presented here it is clear how to improve that study. Instead of the one-component electrodynamic equations, one should use eqs. (19), (20) with two chromodynamic fields generated by the three-color currents. Then, the approach would be more adequate for the quark–gluon plasma.

Let us recapitulate our considerations. Starting from the gauge covariant set of quark plasma kinetic equations we have derived the ideal chromohydrodynamics. We have not specified the collision terms; however, it has been assumed that they have the same properties [eqs. (5) and (7)] as the collision term of the Boltzmann equation. The quark density, hydrodynamic velocity and pressure of the quark plasma are gauge-dependent 3×3 hermitian matrices in the color space. Observing that the distribution functions of local equilibrium provide the quark density and hydrodynamic velocity which commute with each other, one can diagonalize them due to the gauge transformation. Using the diagonal density and velocity matrices one finds the hydrodynamic equations where non-abelian effects disappear.

The author is very grateful to Maria Ekiel-Jeżewska for illuminating discussion on the derivation of hydrodynamics, to Hans-Thomas Elze and Keijo Kajantie for comments and careful reading of the manuscript, and to the Research Institute for Theoretical Physics, University of Helsinki, for kind hospitality.

References

[1] J. Cleymans, R.V. Gavai and E. Suhonen, Phys. Rep. 130 (1986) 217.
[2] F. Low, Phys. Rev. D 12 (1975) 163;
S. Nussinov, Phys. Rev. Lett. 34 (1975) 1286.
[3] B. Andersson, G. Gustafson and T. Sjöstrand, Phys. Rep. 97 (1983) 31.
[4] H. Ehtamo, J. Lindfors and L. McLerran, Z. Phys. C 18 (1983) 341;
T.S. Biro, H.B Nielsen and J. Knoll, Nucl. Phys. B 245 (1984) 449;
A. Białas and W. Czyż, Phys. Rev. D 31 (1985) 198;
A. Kerman, T. Matsui and B. Svetitsky, Phys. Rev. Lett. 56 (1986) 219.
[5] K. Kajantie and T. Matsui, Phys. Lett. B 164 (1985) 373;
G. Gatoff, A.K. Kerman and T. Matsui, Phys. Rev. D 36 (1987) 114;
M. Gyulassy and L. Csernai, report LBL-20610 (Berkeley, 1986), unpublished.
[6] U. Heinz, Phys. Rev. Lett. 51 (1983) 351;
J. Winter, J. Phys. Supp. C 6, 45 (1984) 53;
H.-Th. Elze, M. Gyulassy and D. Vasak, Nucl. Phys. B 276 (1986) 706; Phys. Lett. B 177 (1986) 402.
[7] D.D. Holm and B.A. Kupershmidt, Phys. Lett. A 105 (1984) 225.
[8] St. Mrówczyński and P. Danielewicz, report IPJ/2049/PH (Swierk-Otwock, 1987), unpublished;
St. Mrówczyński, Proc. Workshop on Physics of intermediate and high energy heavy-ion reactions (Cracow, September 1987) (World Scientific, Singapore), to be published.
[9] S.R. de Groot, W.A. van Leeuwen and Ch.G. van Weert, Relativistic kinetic theory (North-Holland, Amsterdam, 1980).