Stability of initial glasma fields

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A system of gluon fields produced in the earliest phase of relativistic heavy-ion collisions, which is called *glasma*, can be described in terms of classical fields. Initially there are chromoelectric and chromomagnetic fields along the collision axis. A linear stability analysis of these fields is performed, assuming that the fields are space-time uniform and using the SU(2) gauge group. We apply Milne coordinates and the gauge condition, which are usually used in studies of glasma. The chromoelectric field is in the Abelian configuration with the corresponding potential linearly depending on coordinates, but the chromomagnetic field is in the non-Abelian configuration generated by the potential of noncommuting components. The chromomagnetic field is found to be unstable, and the growth rate of the unstable mode is derived. Our findings are critically debated and confronted with the numerical simulations by Romatschke and Venugopalan, who found that the evolving glasma is unstable due to the Weibel instability, which is well known in electromagnetic plasma.

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I. INTRODUCTION

The color glass condensate (CGC) approach (see, e.g., the review articles [1,2]) is commonly applied to study the early phase of relativistic heavy-ion collisions at the highest accessible energies at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Within the approach, valence quarks of the colliding nuclei act as color charge sources of long-wavelength chromodynamic fields. Such a system of gluon fields created in the nuclear collision is called *glasma*, and the fields can be approximately treated as classical because of their large occupation numbers. Initially the system is dominated by the chromoelectric and chromomagnetic fields parallel to the beam axis. Later on, transverse fields show up and the system evolves toward thermodynamic equilibrium.

Numerical simulations of the evolving glasma [3,4] showed that the system is unstable. The exponentially growing mode was identified with the Weibel instability [5], also called *filamentation*, which is well-known in the physics of electromagnetic plasma. The relevance of the chromodynamic Weibel instability for quark-gluon plasma produced in relativistic heavy-ion collisions was argued long ago [6] and studied in detail later on; see Ref. [7]. A sufficient condition for the occurrence of the instability is anisotropy of the momentum distribution of plasma charged constituents. They generate a magnetic field, which in turn affects their motion and, as explained in detail in [7], energy is transferred from the particles to the exponentially growing field. The interplay of particles and fields is crucial for the Weibel instability, but, in principle, there are no particles in the glasma—quasiparticles are expected to appear later on as the system evolves towards thermodynamic equilibrium. Nevertheless, high-frequency modes of classical fields are often treated as quasiparticles, and such a treatment led to the interpretation of the instability found in [3,4] as the Weibel mode.

In the early days of quantum chromodynamics (QCD), various configurations of classical chromodynamic fields were found to be unstable; see [8–12]. This observation was a starting point of a whole series of papers [13–17] where the problem of glasma stability was studied. Uniform chromoelectric and chromomagnetic fields along the beam direction were considered, and it was argued [14–17] that the instability found in the simulations [3,4] is not the Weibel but rather Nielsen-Olesen instability [18] of spin 1 charged particles circulating in a uniform chromomagnetic field. There was also considered [16] a possible role of the vacuum instability caused by a strong chromoelectric field which generates particle-antiparticle pairs due to the Schwinger mechanism [19].

An important aspect of the glasma chromomagnetic field was missed in the studies [14–17]. Uniform chromoelectric and chromomagnetic fields are generated by the potentials that are either of single color and depend linearly on coordinates, or by uniform multicolor potentials whose components do not commute with each other. We refer to the former configurations—fully analogous to those known from electrodynamics—as Abelian and the latter ones as non-Abelian. It is important to note that the Abelian and

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non-Abelian configurations are physically nonequivalent as they cannot be related to each other by a gauge transformation [20]. We also note that although the uniform chromomagnetic field is unstable in both Abelian and non-Abelian configurations, the characteristics of the unstable modes are different [21]. We will show in Sec. II that the initial chromoelectric glasma field is in the Abelian configuration, but the chromomagnetic one is the non-Abelian one. In Refs. [14–17], the chromoelectric and chromomagnetic fields were both assumed to be in the Abelian configuration.

We have undertaken an effort to systematically study the stability of classical glasma fields. We have performed [21,22] a linear stability analysis of space-time uniform chromoelectric and chromomagnetic fields in the Abelian configurations, which were studied repeatedly starting with Refs. [10,11], and in the non-Abelian configurations, which were briefly discussed in [12] (see also [23–25]), but not systematically studied. We have derived a complete spectrum of small fluctuations around the background fields which obey the linearized Yang-Mills equations. The spectra of Abelian and non-Abelian configurations are similar but different and they both include unstable modes. We have also discussed parallel chromoelectric and chromomagnetic fields, which occur simultaneously.

The aim of this work is to perform a stability analysis of uniform chromodynamic fields that are relevant for the initial glasma state and confront the results with the simulations [3,4]. We use the Milne—also known as comoving—coordinates, and we apply a specific Fock-Schwinger gauge condition that is usually used in studies of glasma. We note that in our previous works [21,22], we have been using Minkowski coordinates and the background gauge, which is convenient to compare various systems under consideration.

Our paper is organized as follows. In Sec. II we discuss an ansatz [26] that determines a structure of glasma field. We show that the initial chromoelectric field is in the Abelian configuration while the chromomagnetic field is generated by the genuine non-Abelian potential. Yang-Mills equations in Milne coordinates are discussed in Sec. III, where we also derive the equations linearized in a small deviation from the background potential. In Secs. IV and V, a linear stability analysis is performed of the initial chromomagnetic and chromoelectric fields, respectively. The initial glasma fields are treated as stationary, but to check the reliability of the assumption, we consider in Sec. VI the evolution of glasma fields, using the proper time expansion proposed in [27] and developed in [28]. In Sec. VII, our findings are critically debated and discussed in the context of the simulations [3,4,29].

Throughout the paper, we use Minkowski, light-cone, and Milne coordinates

$$(t, \mathbf{x}_\perp, z), \qquad (x^+, \mathbf{x}_\perp, x^-), \qquad (\tau, \mathbf{x}_\perp, \eta), \tag{1}$$

where $x^\pm \equiv (t \pm z)/\sqrt{2}$, $\tau \equiv \sqrt{t^2 - z^2} = \sqrt{2x^+x^-}$, $\eta \equiv \ln(x^+/x^-)/2$, and $\mathbf{x}_\perp = (x, y)$. The indices μ, ν, which label coordinates of spacetime, are $\mu, \nu = t, x, y, z$ in the case of Minkowski coordinates and $\mu, \nu = \tau, x, y, \eta$ in the case of Milne ones. The indices $\alpha, \beta = x, y, z$ and $i, j = x, y$ label, respectively, the Cartesian spatial coordinates and those of the x-y plane, which is transverse to the beam direction along the z-axis. The space-time metric tensor $g_{\mu\nu}$ is diagonal, and the diagonal elements are $(+1, -1, -1, -1)$ in Minkowski and $(+1, -1, -1, -\tau^2)$ in Milne coordinates. The indices $a, b = 1, 2, \dots, N_c^2 - 1$ numerate color components in the adjoint representation of the SU(N_c) gauge group. We omit henceforth the prefix "chromo" when referring to chromoelectric or chromomagnetic fields. Since we study chromodynamics only, this should not be confusing.

II. GLASMA FIELDS

We consider a collision of two heavy ions moving towards each other along the z-axis with the speed of light and colliding at $t = z = 0$. The vector potential of the gluon field is described with the ansatz [26]

$$\begin{aligned}
A^+(x) &= \Theta(x^+)\Theta(x^-)\, x^+\alpha(\tau, \mathbf{x}_\perp),\\
A^-(x) &= -\Theta(x^+)\Theta(x^-)\, x^-\alpha(\tau, \mathbf{x}_\perp),\\
A^i(x) &= \Theta(x^+)\Theta(x^-)\alpha_\perp^i(\tau, \mathbf{x}_\perp) + \Theta(-x^+)\Theta(x^-)\beta_1^i(\mathbf{x}_\perp)\\
&\quad + \Theta(x^+)\Theta(-x^-)\beta_2^i(\mathbf{x}_\perp),
\end{aligned} \tag{2}$$

where the functions $\beta_1^i(x^-, \mathbf{x}_\perp)$ and $\beta_2^i(x^+, \mathbf{x}_\perp)$ represent the precollision potentials, and the functions $\alpha(\tau, \mathbf{x}_\perp)$ and $\alpha_\perp^i(\tau, \mathbf{x}_\perp)$ give the postcollision potentials, $i, j = x, y$.

Components of the potential (2) in the forward light-cone ($x^\pm \geqslant 0$), where glasma is present, are

$$\begin{cases} A^t(x) = z\,\alpha(\tau, \mathbf{x}_\perp),\\ A^z(x) = t\,\alpha(\tau, \mathbf{x}_\perp),\\ A^i(x) = \alpha_\perp^i(\tau, \mathbf{x}_\perp),\end{cases} \qquad \begin{cases} A^\tau(x) = 0,\\ A^\eta(x) = \alpha(\tau, \mathbf{x}_\perp),\\ A^i(x) = \alpha_\perp^i(\tau, \mathbf{x}_\perp),\end{cases} \tag{3}$$

in Minkowski and Milne coordinates, respectively.

The potential (2) satisfies the specific Fock-Schwinger gauge condition, which, written in the Minkowski, light-cone, and Milne coordinate systems, is

$$tA^t - zA^z = 0, \qquad x^-A^+ + x^+A^- = 0, \qquad A^\tau = 0. \tag{4}$$

In the forward light-cone, the vector potential satisfies the sourceless Yang-Mills equations, but the sources enter through the boundary conditions that connect the pre- and postcollision potentials. The conditions read [26]

$$\alpha_\perp^i(0, \mathbf{x}_\perp) = \beta_1^i(\mathbf{x}_\perp) + \beta_2^i(\mathbf{x}_\perp), \tag{5}$$

$$\alpha(0, \mathbf{x}_\perp) = -\frac{ig}{2}\big[\beta_1^i(\mathbf{x}_\perp), \beta_2^i(\mathbf{x}_\perp)\big]. \tag{6}$$

In Minkowski coordinates, the electric and magnetic fields are given as

$$E^\alpha = F^{\alpha t}, \quad B^\alpha = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}F^{\gamma\beta}, \tag{7}$$

where $\alpha, \beta, \gamma = x, y, z$, the strength tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]. \tag{8}$$

and $\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita fully antisymmetric tensor.

Since $t = \tau\cosh\eta$ and $z = \tau\sinh\eta$, the components A^t and A^z given by Eq. (3) vanish at $\tau = 0$, as $\alpha(\tau, \mathbf{x}_\perp)$ is assumed to be regular at $\tau = 0$. However, the derivatives of A^t and A^z with respect to z and t, respectively, are finite at $\tau = 0$. Therefore, the only nonzero components of the electric and magnetic fields at $\tau = 0$ are

$$E(\mathbf{x}_\perp) \equiv E^z(0, \mathbf{x}_\perp) = -2\alpha(0, \mathbf{x}_\perp), \tag{9}$$

$$B(\mathbf{x}_\perp) \equiv B^z(0, \mathbf{x}_\perp) = -\partial_y \alpha_\perp^x(0, \mathbf{x}_\perp) + \partial_x \alpha_\perp^y(0, \mathbf{x}_\perp) - ig[\alpha_\perp^y(0, \mathbf{x}_\perp), \alpha_\perp^x(0, \mathbf{x}_\perp)]. \tag{10}$$

When the precollision potentials β_1^i and β_2^i are uniform, that is, independent of $\mathbf{x}_\perp$, the glasma initial potential is

$$\begin{aligned} A_a^t &= z\,\alpha_a = \frac{g}{2}\, z\, f^{abc} \beta_{1b}^i \beta_{2c}^i, \\ A_a^z &= t\,\alpha_a = \frac{g}{2}\, t\, f^{abc} \beta_{1b}^i \beta_{2c}^i, \\ A_a^i &= \alpha_{\perp a}^i = \beta_{1a}^i + \beta_{2a}^i, \end{aligned} \tag{11}$$

which is written in the adjoint representation of the SU(N_c) gauge group. The potential generates the initial electric and magnetic fields, which are

$$E_a = -g f^{abc} \beta_{1b}^i \beta_{2c}^i, \tag{12}$$

$$B_a = -g \epsilon^{zij} f^{abc} \beta_{1b}^i \beta_{2c}^j. \tag{13}$$

Equations (11) clearly show that the initial uniform electric field (12) is in an Abelian configuration, that is, it is generated by the potential components A^t and A^z linearly depending on t and z, while the magnetic field (13) is in a non-Abelian configuration, that is, it is generated by the uniform and non-commuting potential components A^x and A^y.

One shows that the Abelian and non-Abelian configurations of the same uniform electric or magnetic field are physically nonequivalent by observing that the potential of the Abelian configuration satisfies the Yang-Mills equations with vanishing color current, while the potential of the non-Abelian configuration can solve the Yang-Mills equations only with an appropriately chosen color current. Since a nonzero current cannot be nulled by gauge transformation, the Abelian and non-Abelian configurations are nonequivalent. We will return to this issue in Sec. IV.

III. LINEARIZED YANG-MILLS EQUATIONS IN MILNE COORDINATES

In curvilinear coordinates, the Yang-Mills equations are

$$\nabla_\mu F^{\mu\nu} = j^\nu, \tag{14}$$

where ∇_μ is the covariant derivative, which includes Christoffel symbols and the gauge potential,

$$F^{\mu\nu} \equiv \nabla^\mu A^\nu - \nabla^\nu A^\mu, \tag{15}$$

and j^μ is the color current. In the adjoint representation of SU(N_c) group, the covariant derivative acts on the vector potential A_a^μ as

$$\nabla_\mu^{ab} A_b^\nu \equiv \partial_\mu A_a^\nu + \Gamma_{\mu\rho}^\nu A_a^\rho + g f^{abc} A_\mu^b A_c^\nu, \tag{16}$$

where $\Gamma_{\mu\rho}^\nu$ is the Christoffel symbol. We note that upper and lower color indices are not distinguished from each other. Since $\Gamma_{\nu\rho}^\mu = \Gamma_{\rho\nu}^\mu$, the strength tensor $F_{\mu\nu}^a$ equals

$$F_{\mu\nu}^a = (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{17}$$

To derive an explicit form of the Yang-Mills equations in Milne coordinates, we use the formula

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-\bar{g}}} D_\mu (\sqrt{-\bar{g}}\, F^{\mu\nu}), \tag{18}$$

where $\bar{g}$ is the determinant of a metric tensor, which in the case of Milne coordinates equals $\bar{g} = -\tau^2$, and D_μ is the covariant derivative, which includes the gauge potential but no Christoffel symbol. The Yang-Mills equations can be written as

$$(\nabla_\mu F^{\mu\nu})_a = \frac{1}{\tau}\, \partial_\mu \big(\tau\, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^a\big) + g g^{\mu\rho} g^{\nu\sigma} f^{abc} A_\mu^b F_{\rho\sigma}^c = j_a^\nu, \tag{19}$$

where the derivatives ∂_μ but not ∂^μ are used, and the strength tensor is given by Eq. (17), which does not include Christoffel symbols.

Since the Yang-Mills equations play a central role in our analysis, we write them explicitly in the case of the SU(2) gauge group, when $f^{abc} = \epsilon^{abc}$, and the gauge condition is $A^\tau = 0$. The equations read

$$-\partial_x \partial_\tau A_a^x - \partial_y \partial_\tau A_a^y - \frac{1}{\tau^2}\, \partial_\eta \partial_\tau \big(\tau^2 A_a^\eta\big) + g\epsilon^{abc} A_b^x \partial_\tau A_c^x + g\epsilon^{abc} A_b^y \partial_\tau A_c^y + g\epsilon^{abc} A_b^\eta \partial_\tau \big(\tau^2 A_c^\eta\big) = j_a^\tau, \tag{20}$$

$$\begin{aligned} &\partial_\tau^2 A_a^x + \frac{1}{\tau}\, \partial_\tau A_a^x - \partial_y \big(\partial_y A_a^x - \partial_x A_a^y - g\epsilon^{abc} A_b^y A_c^x\big) - \frac{1}{\tau^2}\, \partial_\eta \big(\partial_\eta A_a^x - \tau^2 \partial_x A_a^\eta - g\tau^2 \epsilon^{abc} A_b^\eta A_c^x\big) \\ &+ g\epsilon^{abc} A_b^y \big(\partial_y A_c^x - \partial_x A_c^y\big) + g\epsilon^{abc} A_b^\eta \big(\partial_\eta A_c^x - \tau^2 \partial_x A_c^\eta\big) - g^2 \big(A_b^y A_a^y A_b^x - A_b^y A_b^y A_a^x + \tau^2 A_b^\eta A_a^\eta A_b^x - \tau^2 A_b^\eta A_b^\eta A_a^x\big) = j_a^x, \end{aligned} \tag{21}$$

$$\begin{aligned} &\partial_\tau^2 A_a^y + \frac{1}{\tau}\, \partial_\tau A_a^y - \partial_x \big(\partial_x A_a^y - \partial_y A_a^x - g\epsilon^{abc} A_b^x A_c^y\big) - \frac{1}{\tau^2}\, \partial_\eta \big(\partial_\eta A_a^y - \tau^2 \partial_y A_a^\eta - g\tau^2 \epsilon^{abc} A_b^\eta A_c^y\big) + g\epsilon^{abc} A_b^x \big(\partial_x A_c^y - \partial_y A_c^x\big) \\ &+ g\epsilon^{abc} A_b^\eta \big(\partial_\eta A_c^y - \tau^2 \partial_y A_c^\eta\big) - g^2 \big(A_b^x A_a^x A_b^y - A_b^x A_b^x A_a^y + \tau^2 A_b^\eta A_a^\eta A_b^y - \tau^2 A_b^\eta A_b^\eta A_a^y\big) = j_a^y, \end{aligned} \tag{22}$$

$$\begin{aligned} &\partial_\tau^2 \big(\tau^2 A_a^\eta\big) - \frac{1}{\tau} \partial_\tau \big(\tau^2 A_a^\eta\big) - \partial_x \big(\tau^2 \partial_x A_a^\eta - \partial_\eta A_a^x - g\tau^2 \epsilon^{abc} A_b^x A_c^\eta\big) - \partial_y \big(\tau^2 \partial_y A_a^\eta - \partial_\eta A_a^y - g\tau^2 \epsilon^{abc} A_b^y A_c^\eta\big) \\ &+ g\epsilon^{abc} A_b^x \big(\tau^2 \partial_x A_c^\eta - \partial_\eta A_c^x\big) + g\epsilon^{abc} A_b^y \big(\tau^2 \partial_y A_c^\eta - \partial_\eta A_c^y\big) + g^2 \tau^2 \big(- A_b^x A_a^x A_b^\eta + A_b^x A_b^x A_a^\eta - A_b^y A_a^y A_b^\eta + A_b^y A_b^y A_a^\eta\big) = \tau^2 j_a^\eta. \end{aligned} \tag{23}$$

Now, we assume that the background potential $\bar{A}^\mu_a$ solves the Yang-Mills equations, and we consider small fluctuations a^μ_a around $\bar{A}^\mu_a$. So, we define the potential

$$A^\mu_a(\tau, x, y, \eta) \equiv \bar{A}^\mu_a(\tau, x, y, \eta) + a^\mu_a(\tau, x, y, \eta), \tag{24}$$

such that $|\bar{A}^\mu_a(\tau, x, y, \eta)| \gg |a^\mu_a(\tau, x, y, \eta)|$. Substituting the potential (24) into the Yang-Mills equations (20)–(23) and neglecting the terms nonlinear in a^μ_a, one finds

$$-\partial_x\partial_\tau a^x_a - \partial_y\partial_\tau a^y_a - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a^\eta_a\big) + g\epsilon^{abc}\big(\bar{A}^x_b\partial_\tau a^x_c + a^x_b\partial_\tau\bar{A}^x_c + \bar{A}^y_b\partial_\tau a^y_c + a^y_b\partial_\tau\bar{A}^y_c + \bar{A}^\eta_b\partial_\tau(\tau^2 a^\eta_c) + a^\eta_b\partial_\tau(\tau^2\bar{A}^\eta_c)\big) = 0, \tag{25}$$

$$\begin{aligned}
&\partial^2_\tau a^x_a + \frac{1}{\tau}\,\partial_\tau a^x_a - \partial_y\big(\partial_y a^x_a - \partial_x a^y_a - g\epsilon^{abc}\big(\bar{A}^y_b a^x_c + a^y_b\bar{A}^x_c\big)\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a^x_a - \tau^2\partial_x a^\eta_a - g\tau^2\epsilon^{abc}\big(\bar{A}^\eta_b a^x_c + a^\eta_b\bar{A}^x_c\big)\big)\\
&+ g\epsilon^{abc}\big(\bar{A}^y_b\big(\partial_y a^x_c - \partial_x a^y_c\big) + a^y_b\big(\partial_y\bar{A}^x_c - \partial_x\bar{A}^y_c\big) + \bar{A}^\eta_b\big(\partial_\eta a^x_c - \tau^2\partial_x a^\eta_c\big) + a^\eta_b\big(\partial_\eta\bar{A}^x_c - \tau^2\partial_x\bar{A}^\eta_c\big)\big)\\
&- g^2\big(\bar{A}^y_b\bar{A}^y_a a^x_b + \bar{A}^y_b a^y_a\bar{A}^x_b + a^y_b\bar{A}^y_a\bar{A}^x_b - \bar{A}^y_b\bar{A}^y_b a^x_a - \bar{A}^y_b a^y_b\bar{A}^x_a - a^y_b\bar{A}^y_b\bar{A}^x_a\\
&+ \tau^2\big(\bar{A}^\eta_b\bar{A}^\eta_a a^x_b + \bar{A}^\eta_b a^\eta_a\bar{A}^x_b + a^\eta_b\bar{A}^\eta_a\bar{A}^x_b - \bar{A}^\eta_b\bar{A}^\eta_b a^x_a - \bar{A}^\eta_b a^\eta_b\bar{A}^x_a - a^\eta_b\bar{A}^\eta_b\bar{A}^x_a\big)\big) = 0,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\partial^2_\tau a^y_a + \frac{1}{\tau}\,\partial_\tau a^y_a - \partial_x\big(\partial_x a^y_a - \partial_y a^x_a - g\epsilon^{abc}\big(\bar{A}^x_b a^y_c + a^x_b\bar{A}^y_c\big)\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a^y_a - \tau^2\partial_y a^\eta_a - g\tau^2\epsilon^{abc}\big(\bar{A}^\eta_b a^y_c + a^\eta_b\bar{A}^y_c\big)\big)\\
&+ g\epsilon^{abc}\big(\bar{A}^x_b\big(\partial_x a^y_c - \partial_y a^x_c\big) + a^x_b\big(\partial_x\bar{A}^y_c - \partial_y\bar{A}^x_c\big) + \bar{A}^\eta_b\big(\partial_\eta a^y_c - \tau^2\partial_y a^\eta_c\big) + a^\eta_b\big(\partial_\eta\bar{A}^y_c - \tau^2\partial_y\bar{A}^\eta_c\big)\big)\\
&- g^2\big(\bar{A}^x_b\bar{A}^x_a a^y_b + \bar{A}^x_b a^x_a\bar{A}^y_b + a^x_b\bar{A}^x_a\bar{A}^y_b - \bar{A}^x_b\bar{A}^x_b a^y_a - \bar{A}^x_b a^x_b\bar{A}^y_a - a^x_b\bar{A}^x_b\bar{A}^y_a\\
&+ \tau^2\big(\bar{A}^\eta_b\bar{A}^\eta_a a^y_b + \bar{A}^\eta_b a^\eta_a\bar{A}^y_b + a^\eta_b\bar{A}^\eta_a\bar{A}^y_b - \bar{A}^\eta_b\bar{A}^\eta_b a^y_a - \bar{A}^\eta_b a^\eta_b\bar{A}^y_a - a^\eta_b\bar{A}^\eta_b\bar{A}^y_a\big)\big) = 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\partial^2_\tau\big(\tau^2 a^\eta_a\big) - \frac{1}{\tau}\partial_\tau\big(\tau^2 a^\eta_a\big) - \partial_x\big(\tau^2\partial_x a^\eta_a - \partial_\eta a^x_a - g\tau^2\epsilon^{abc}\big(\bar{A}^x_b a^\eta_c + a^x_b\bar{A}^\eta_c\big)\big) - \partial_y\big(\tau^2\partial_y a^\eta_a - \partial_\eta a^y_a - g\tau^2\epsilon^{abc}\big(\bar{A}^y_b a^\eta_c + a^y_b\bar{A}^\eta_c\big)\big)\\
&+ g\epsilon^{abc}\big(\bar{A}^x_b\big(\tau^2\partial_x a^\eta_c - \partial_\eta a^x_c\big) + a^x_b\big(\tau^2\partial_x\bar{A}^\eta_c - \partial_\eta\bar{A}^x_c\big) + \bar{A}^y_b\big(\tau^2\partial_y a^\eta_c - \partial_\eta a^y_c\big) + a^y_b\big(\tau^2\partial_y\bar{A}^\eta_c - \partial_\eta\bar{A}^y_c\big)\big)\\
&+ g^2\tau^2\big(-\bar{A}^x_b\bar{A}^x_a a^\eta_b - \bar{A}^x_b a^x_a\bar{A}^\eta_b - a^x_b\bar{A}^x_a\bar{A}^\eta_b + \bar{A}^x_b\bar{A}^x_b a^\eta_a + \bar{A}^x_b a^x_b\bar{A}^\eta_a + a^x_b\bar{A}^x_b\bar{A}^\eta_a\\
&- \bar{A}^y_b\bar{A}^y_a a^\eta_b - \bar{A}^y_b a^y_a\bar{A}^\eta_b - a^y_b\bar{A}^y_a\bar{A}^\eta_b + \bar{A}^y_b\bar{A}^y_b a^\eta_a + \bar{A}^y_b a^y_b\bar{A}^\eta_a + a^y_b\bar{A}^y_b\bar{A}^\eta_a\big) = 0.
\end{aligned} \tag{28}$$

IV. STABILITY OF MAGNETIC FIELD

We consider here the stability of the magnetic field B generated along the z-axis at the earliest phase of a heavy-ion collision. The field is given by the formula (13), that is, it is assumed to be space-time uniform. The validity of the assumption is discussed in Sec. VII.

The field occurs due to the precollision potentials β^i_{1a} and β^i_{2a}, which are independent of each other. The two potentials are parametrized by means of two parameters λ and B in the following way:

$$\beta^i_{1a} = \delta^{ix}\delta^{a3}\frac{1}{\lambda}\sqrt{\frac{B}{g}}, \quad \beta^i_{2a} = \delta^{iy}\delta^{a2}\lambda\sqrt{\frac{B}{g}}. \tag{29}$$

The corresponding background four-potential can be written in matrix notation as

$$\bar{A}^\mu_a = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \lambda\sqrt{\frac{B}{g}} & 0\\ 0 & \frac{1}{\lambda}\sqrt{\frac{B}{g}} & 0 & 0\end{bmatrix}, \tag{30}$$

where the Lorentz index μ numerates the columns, and the color index a numerates the rows. The potential (30) substituted into Eqs. (12) and (13) gives

$$E^\alpha_a = 0, \quad B^\alpha_a = \delta^{\alpha z}\delta^{a1}B. \tag{31}$$

So, we have the vanishing electric field and uniform magnetic field of color 1 along the z-axis.

The magnetic field (31) is independent of λ but the configurations of different lambdas are known to be physically nonequivalent [20]. This is easily demonstrated substituting the background potential (30) into the Yang-Mills equations (20)–(23). Then, one finds

$$\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\lambda}\sqrt{gB^3} & 0\\ 0 & \lambda\sqrt{gB^3} & 0 & 0\end{bmatrix} = j^\nu_a. \tag{32}$$

The current (32) squared, which is gauge invariant, equals

$$j^\mu_a\, j_{a\,\mu} = -\left(\frac{1}{\lambda^2} + \lambda^2\right)gB^3, \tag{33}$$

and it depends on $(\lambda^{-2} + \lambda^2)$. Therefore, the potential configurations (30) of different $(\lambda^{-2} + \lambda^2)$ are gauge nonequivalent [20], nevertheless they produce the same field strength and the same energy density.

Equation (32) also shows that, in contrast to the Abelian configuration of uniform magnetic field, the non-Abelian one (30) does not solve the Yang-Mills equations with vanishing current. It means that the field must evolve in time to satisfy the equations. Consequently, a stationary character of the background field is only an approximation which is applicable in the stability analysis if the rate of change of the background field is much smaller than the rate of change of small fluctuations. We return to this issue in Secs. VI and VII.

Performing the stability analysis, we temporarily assume following [12] that the current, which equals the left-hand side of Eq. (32), enters the Yang-Mills equation. Then, the stability analysis is performed in a standard way: the background potential (30) solves the equations of motion, and one checks whether small perturbations of this stationary solution grow or decay.

With the background potential (30), the linearized Yang-Mills Eqs. (25)–(28) split into colors are

$$-\partial_x\partial_\tau a_1^x - \partial_y\partial_\tau a_1^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_1^\eta\big) + g\big(-\bar{A}_3^x\partial_\tau a_2^x + \bar{A}_2^y\partial_\tau a_3^y\big) = 0, \tag{34}$$

$$-\partial_x\partial_\tau a_2^x - \partial_y\partial_\tau a_2^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_2^\eta\big) + g\bar{A}_3^x\partial_\tau a_1^x = 0, \tag{35}$$

$$-\partial_x\partial_\tau a_3^x - \partial_y\partial_\tau a_3^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_3^\eta\big) - g\bar{A}_2^y\partial_\tau a_1^y = 0, \tag{36}$$

$$\partial_\tau^2 a_1^x + \frac{1}{\tau}\,\partial_\tau a_1^x - \partial_y\big(\partial_y a_1^x - \partial_x a_1^y - g\big(\bar{A}_2^y a_3^x + a_2^y\bar{A}_3^x\big)\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_1^x - \tau^2\partial_x a_1^\eta - g\tau^2 a_2^\eta\bar{A}_3^x\big) + g\bar{A}_2^y\big(\partial_y a_3^x - \partial_x a_3^y\big) + g^2\bar{A}_2^y\bar{A}_2^y a_1^x = 0, \tag{37}$$

$$\partial_\tau^2 a_2^x + \frac{1}{\tau}\,\partial_\tau a_2^x - \partial_y\big(\partial_y a_2^x - \partial_x a_2^y + g a_1^y\bar{A}_3^x\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_2^x - \tau^2\partial_x a_2^\eta + g\tau^2 a_1^\eta\bar{A}_3^x\big) - g^2 a_3^y\bar{A}_2^y\bar{A}_3^x = 0, \tag{38}$$

$$\partial_\tau^2 a_3^x + \frac{1}{\tau}\,\partial_\tau a_3^x - \partial_y\big(\partial_y a_3^x - \partial_x a_3^y + g\bar{A}_2^y a_1^x\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_3^x - \tau^2\partial_x a_3^\eta\big) - g\bar{A}_2^y\big(\partial_y a_1^x - \partial_x a_1^y\big) + g^2\big(\bar{A}_2^y\bar{A}_2^y a_3^x + 2a_2^y\bar{A}_2^y\bar{A}_3^x\big) = 0, \tag{39}$$

$$\partial_\tau^2 a_1^y + \frac{1}{\tau}\,\partial_\tau a_1^y - \partial_x\big(\partial_x a_1^y - \partial_y a_1^x + g\big(\bar{A}_3^x a_2^y + a_3^x\bar{A}_2^y\big)\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_1^y - \tau^2\partial_y a_1^\eta + g\tau^2 a_3^\eta\bar{A}_2^y\big) - g\bar{A}_3^x\big(\partial_x a_2^y - \partial_y a_2^x\big) + g^2\bar{A}_3^x\bar{A}_3^x a_1^y = 0, \tag{40}$$

$$\partial_\tau^2 a_2^y + \frac{1}{\tau}\,\partial_\tau a_2^y - \partial_x\big(\partial_x a_2^y - \partial_y a_2^x - g\bar{A}_3^x a_1^y\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_2^y - \tau^2\partial_y a_2^\eta\big) + g\bar{A}_3^x\big(\partial_x a_1^y - \partial_y a_1^x\big) + g^2\big(\bar{A}_3^x\bar{A}_3^x a_2^y + 2a_3^x\bar{A}_3^x\bar{A}_2^y\big) = 0, \tag{41}$$

$$\partial_\tau^2 a_3^y + \frac{1}{\tau}\,\partial_\tau a_3^y - \partial_x\big(\partial_x a_3^y - \partial_y a_3^x - g a_1^x\bar{A}_2^y\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_3^y - \tau^2\partial_y a_3^\eta - g\tau^2 a_1^\eta\bar{A}_2^y\big) - g^2 a_2^x\bar{A}_3^x\bar{A}_2^y = 0, \tag{42}$$

$$\begin{aligned}&\partial_\tau^2\big(\tau^2 a_1^\eta\big) - \frac{1}{\tau}\partial_\tau\big(\tau^2 a_1^\eta\big) - \partial_x\big(\tau^2\partial_x a_1^\eta - \partial_\eta a_1^x + g\tau^2\bar{A}_3^x a_2^\eta\big) - \partial_y\big(\tau^2\partial_y a_1^\eta - \partial_\eta a_1^y - g\tau^2\bar{A}_2^y a_3^\eta\big)\\ &\quad - g\big(\bar{A}_3^x\big(\tau^2\partial_x a_2^\eta - \partial_\eta a_2^x\big) - \bar{A}_2^y\big(\tau^2\partial_y a_3^\eta - \partial_\eta a_3^y\big)\big) + g^2\tau^2\big(\bar{A}_3^x\bar{A}_3^x + \bar{A}_2^y\bar{A}_2^y\big)a_1^\eta = 0,\end{aligned} \tag{43}$$

$$\partial_\tau^2\big(\tau^2 a_2^\eta\big) - \frac{1}{\tau}\partial_\tau\big(\tau^2 a_2^\eta\big) - \partial_x\big(\tau^2\partial_x a_2^\eta - \partial_\eta a_2^x - g\tau^2\bar{A}_3^x a_1^\eta\big) - \partial_y\big(\tau^2\partial_y a_2^\eta - \partial_\eta a_2^y\big) + g\bar{A}_3^x\big(\tau^2\partial_x a_1^\eta - \partial_\eta a_1^x\big) + g^2\tau^2\bar{A}_3^x\bar{A}_3^x a_2^\eta = 0, \tag{44}$$

$$\partial_\tau^2\big(\tau^2 a_3^\eta\big) - \frac{1}{\tau}\partial_\tau\big(\tau^2 a_3^\eta\big) - \partial_x\big(\tau^2\partial_x a_3^\eta - \partial_\eta a_3^x\big) - \partial_y\big(\tau^2\partial_y a_3^\eta - \partial_\eta a_3^y + g\tau^2\bar{A}_2^y a_1^\eta\big) - g\bar{A}_2^y\big(\tau^2\partial_y a_1^\eta - \partial_\eta a_1^y\big) + g^2\tau^2\bar{A}_2^y\bar{A}_2^y a_3^\eta = 0. \tag{45}$$

So, we have a system of 12 equations to be solved. We have managed to solve exactly the analogous set of equations using the Minkowski coordinates, background gauge, and $\lambda = 1$ condition [21]. Thanks to Minkowski coordinates, the problem was fully algebraic after the Fourier transformation of space and time coordinates. Due to the background gauge, a mixing of various color components was minimal, and the condition $\lambda = 1$ provided an axial symmetry with respect to the axis along the magnetic field. Here we deal with a more complicated system of equations. So, we consider a simplified situation when an evolution of longitudinal and transverse potential components is treated separately. Specifically, we discuss two special cases that still allow one to reveal characteristic features of the problem.

A. Special case: $a_a^\eta = 0$ & $a_a^x \neq 0$, $a_a^y \neq 0$

When $a_a^\eta = 0$, Eqs. (34)–(45) read

$$-\partial_x\partial_\tau a_1^x - \partial_y\partial_\tau a_1^y - g\big(\bar{A}_3^x\partial_\tau a_2^x - \bar{A}_2^y\partial_\tau a_3^y\big) = 0, \tag{46}$$

$$- \partial_x \partial_\tau a_2^x - \partial_y \partial_\tau a_2^y + g\bar{A}_3^x \partial_\tau a_1^x = 0, \tag{47}$$

$$- \partial_x \partial_\tau a_3^x - \partial_y \partial_\tau a_3^y - g\bar{A}_2^y \partial_\tau a_1^y = 0, \tag{48}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_1^x + \partial_x\partial_y a_1^y + g\bar{A}_3^x \partial_y a_2^y + g\bar{A}_2^y\big(2\partial_y a_3^x - \partial_x a_3^y\big) + g^2 \bar{A}_2^y \bar{A}_2^y a_1^x = 0, \tag{49}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_2^x + \partial_x\partial_y a_2^y - g\bar{A}_3^x \partial_y a_1^y - g^2 \bar{A}_2^y \bar{A}_3^x a_3^y = 0, \tag{50}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_3^x + \partial_x\partial_y a_3^y - g\bar{A}_2^y\big(2\partial_y a_1^x - \partial_x a_1^y\big) + g^2\big(\bar{A}_2^y \bar{A}_2^y a_3^x + 2\bar{A}_2^y \bar{A}_3^x a_2^y\big) = 0, \tag{51}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_1^y + \partial_y\partial_x a_1^x - g\bar{A}_2^y \partial_x a_3^x - g\bar{A}_3^x\big(2\partial_x a_2^y - \partial_y a_2^x\big) + g^2 \bar{A}_3^x \bar{A}_3^x a_1^y = 0, \tag{52}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_2^y + \partial_y\partial_x a_2^x + g\bar{A}_3^x\big(2\partial_x a_1^y - \partial_y a_1^x\big) + g^2\big(\bar{A}_3^x \bar{A}_3^x a_2^y + 2\bar{A}_3^x \bar{A}_2^y a_3^x\big) = 0, \tag{53}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \frac{1}{\tau^2}\,\partial_\eta^2\right)a_3^y + \partial_y\partial_x a_3^x + g\bar{A}_2^y \partial_x a_1^x - g^2 \bar{A}_3^x \bar{A}_2^y a_2^x = 0, \tag{54}$$

$$\partial_x\partial_\eta a_1^x + \partial_y\partial_\eta a_1^y + g\big(\bar{A}_3^x \partial_\eta a_2^x - \bar{A}_2^y \partial_\eta a_3^y\big) = 0, \tag{55}$$

$$\partial_x\partial_\eta a_2^x + \partial_y\partial_\eta a_2^y - g\bar{A}_3^x \partial_\eta a_1^x = 0, \tag{56}$$

$$\partial_x\partial_\eta a_3^x + \partial_y\partial_\eta a_3^y + g\bar{A}_2^y \partial_\eta a_1^y = 0. \tag{57}$$

One sees that Eqs. (46)–(48) and (55)–(57) are solved if

$$\partial_x a_1^x + \partial_y a_1^y + g\big(\bar{A}_3^x a_2^x - \bar{A}_2^y a_3^y\big) = 0, \tag{58}$$

$$\partial_x a_2^x + \partial_y a_2^y - g\bar{A}_3^x a_1^x = 0, \tag{59}$$

$$\partial_x a_3^x + \partial_y a_3^y + g\bar{A}_2^y a_1^y = 0. \tag{60}$$

Substituting $\partial_y a_1^y = -\partial_x a_1^x - g(\bar{A}_3^x a_2^x - \bar{A}_2^y a_3^y)$ and $\partial_y a_2^y = -\partial_x a_2^x + g\bar{A}_3^x a_1^x$ into Eqs. (49) and (50), and $\partial_y a_3^y = -\partial_x a_3^x - g\bar{A}_2^y a_1^y$ into Eq. (51), one finds

$$\big(\Box + g^2\big(\bar{A}_3^x \bar{A}_3^x + \bar{A}_2^y \bar{A}_2^y\big)\big)a_1^x - 2g\bar{A}_3^x \partial_x a_2^x + 2g\bar{A}_2^y \partial_y a_3^x = 0, \tag{61}$$

$$\big(\Box + g^2 \bar{A}_3^x \bar{A}_3^x\big)a_2^x + 2g\bar{A}_3^x \partial_x a_1^x - 2g^2 \bar{A}_3^x \bar{A}_2^y a_3^y = 0, \tag{62}$$

$$\big(\Box + g^2 \bar{A}_2^y \bar{A}_2^y\big)a_3^x - 2g\bar{A}_2^y \partial_y a_1^x + 2g^2 \bar{A}_2^y \bar{A}_3^x a_2^y = 0, \tag{63}$$

where

$$\Box \equiv \partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2. \tag{64}$$

Proceeding analogously with Eqs. (52), (53), and (54), we obtain

$$\big(\Box + g^2\big(\bar{A}_3^x \bar{A}_3^x + \bar{A}_2^y \bar{A}_2^y\big)\big)a_1^y + 2g\bar{A}_2^y \partial_y a_3^y - 2g\bar{A}_3^x \partial_x a_2^y = 0, \tag{65}$$

$$\big(\Box + g^2 \bar{A}_3^x \bar{A}_3^x\big)a_2^y + 2g\bar{A}_3^x \partial_x a_1^y + 2g^2 \bar{A}_3^x \bar{A}_2^y a_3^x = 0, \tag{66}$$

$$\big(\Box + g^2 \bar{A}_2^y \bar{A}_2^y\big)a_3^y - 2g\bar{A}_2^y \partial_y a_1^y - 2g^2 \bar{A}_2^y \bar{A}_3^x a_2^x = 0. \tag{67}$$

Defining the functions

$$V_a^\pm \equiv a_a^x \pm i a_a^y, \tag{68}$$

Equations (61)–(67) provide

$$\big(\Box + g^2\big(\bar{A}_3^x \bar{A}_3^x + \bar{A}_2^y \bar{A}_2^y\big)\big)V_1^\pm - 2g\bar{A}_3^x \partial_x V_2^\pm + 2g\bar{A}_2^y \partial_y V_3^\pm = 0, \tag{69}$$

$$\big(\Box + g^2 \bar{A}_3^x \bar{A}_3^x\big)V_2^\pm + 2g\bar{A}_3^x \partial_x V_1^\pm \pm 2ig^2 \bar{A}_3^x \bar{A}_2^y V_3^\pm = 0, \tag{70}$$

$$\big(\Box + g^2 \bar{A}_2^y \bar{A}_2^y\big)V_3^\pm - 2g\bar{A}_2^y \partial_y V_1^\pm \mp 2ig^2 \bar{A}_2^y \bar{A}_3^x V_2^\pm = 0. \tag{71}$$

One sees that the equations of V_1^+, V_2^+, V_3^+ and those of V_1^-, V_2^-, V_3^- form closed systems, that is, the functions V_a^+ do not mix up with V_a^-.

Keeping in mind that $\bar{A}_2^y = \lambda\sqrt{B/g}$ and $\bar{A}_3^x = \lambda^{-1}\sqrt{B/g}$, Eqs. (69), (70), and (71) can be rewritten in the matrix notation as

$$\begin{bmatrix} \Box + (\lambda^2+\lambda^{-2})gB & -2i\lambda^{-1}\sqrt{gB}\,k_x & 2i\lambda\sqrt{gB}\,k_y \\ 2i\lambda^{-1}\sqrt{gB}\,k_x & \Box + \lambda^{-2}gB & \pm 2igB \\ -2i\lambda\sqrt{gB}\,k_y & \mp 2igB & \Box + \lambda^2 gB \end{bmatrix} \begin{bmatrix} V_1^\pm \\ V_2^\pm \\ V_3^\pm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{72}$$

where we have assumed that the functions $V_a^\pm$ depend on x, y, η through $e^{i(k_x x + k_y y + \nu\eta)}$ and consequently the d'Alembertian is redefined as

$$\Box \equiv \partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_x^2 + k_y^2 + \frac{\nu^2}{\tau^2}. \tag{73}$$

The eigenvalues Λ of the matrix in Eq. (72) are provided by the cubic equation

$$\det\begin{bmatrix} \Box + (\lambda^2+\lambda^{-2})gB - \Lambda & -2i\lambda^{-1}\sqrt{gB}\,k_x & 2i\lambda\sqrt{gB}\,k_y \\ 2i\lambda^{-1}\sqrt{gB}\,k_x & \Box + \lambda^{-2}gB - \Lambda & \pm 2igB \\ -2i\lambda\sqrt{gB}\,k_y & \mp 2igB & \Box + \lambda^2 gB - \Lambda \end{bmatrix} = 0, \tag{74}$$

which gives

$$\begin{aligned} &(\Box + (\lambda^2+\lambda^{-2})gB - \Lambda)(\Box + \lambda^{-2}gB - \Lambda)(\Box + \lambda^2 gB - \Lambda) \\ &-4g^2B^2(\Box + (\lambda^2+\lambda^{-2})gB - \Lambda) - 4\lambda^{-2}gB\,k_x^2(\Box + \lambda^2 gB - \Lambda) - 4\lambda^2 gB\,k_y^2(\Box + \lambda^{-2}gB - \Lambda) = 0. \end{aligned} \tag{75}$$

One sees that Eq. (75) is the same for the equation of V^+ and V^-. Consequently, the eigenvalues of the matrices, which enter equations of V^+ and V^-, are the same.

When $k_x = 0$ and $k_y = 0$, Eq. (75) becomes

$$(\Box + (\lambda^2+\lambda^{-2})gB - \Lambda^{(0)})(\Lambda^2 - (2\Box + (\lambda^{-2}+\lambda^2)gB)\Lambda^{(0)} + (\Box + \lambda^{-2}gB)(\Box + \lambda^2 gB) - 4g^2B^2) = 0, \tag{76}$$

and it is solved by

$$\Lambda_1 = \Box + (\lambda^2+\lambda^{-2})gB, \quad \Lambda_\pm = \Box + \tfrac{1}{2}(\lambda^{-2}+\lambda^2 \pm \sqrt{(\lambda^{-2}+\lambda^2)^2 + 12})gB. \tag{77}$$

After the diagonalization of the matrix equation (72), one gets three equations:

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} + b_1^2\right) f_1(\tau) = 0, \tag{78}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} \pm b_\pm^2\right) f_\pm(\tau) = 0, \tag{79}$$

where

$$b_1^2 \equiv (\lambda^2+\lambda^{-2})gB, \tag{80}$$

$$b_\pm^2 \equiv \tfrac{1}{2}(\sqrt{(\lambda^{-2}+\lambda^2)^2 + 12} \pm (\lambda^{-2}+\lambda^2))gB, \tag{81}$$

and the functions f_1 and $f_\pm$ are linear combinations of either V_1^+, V_2^+, V_3^+ or V_1^-, V_2^-, V_3^-. When $\lambda = 1$, we have

$$b_1^2 = 2gB, \quad b_+^2 = 3gB, \quad b_-^2 = gB. \tag{82}$$

One sees that the equations of f_1 and f_+ are the Bessel equation and that of f_- is the modified Bessel equation. The equations are briefly discussed for the reader's convenience in the Appendix. Since we are interested in solutions, which are everywhere finite, as we study small fluctuations around the background field, the solutions $f_1(\tau)$ and $f_+(\tau)$ are the Bessel function of imaginary order $J_{i\nu}(b_1\tau)$ or $J_{i\nu}(b_+\tau)$, and $f_-(\tau)$ is the modified Bessel function $I_{i\nu}(b_-\tau)$. While the solutions $f_1(\tau)$ and $f_+(\tau)$ are oscillatory, the solution $f_-(\tau)$ grows as $\exp(\sqrt{b_- gB}\,\tau)$ when $\tau^2 > \nu^2/b_-^2$ and represents the instability analogous to the Nielsen-Olesen instability [18]. The difference is that the unstable Nielsen-Olesen mode appears in the Abelian configuration of the magnetic background field. We note that when $\lambda = 1$, the equation (79) of f_- fully coincides with that of the unstable mode of the Abelian configuration after a dependence on transverse coordinates is separated out using the Hamiltonian of the harmonic oscillator [16]. The time dependence of the unstable solution of Abelian configuration at any k_x and k_y coincides with that of the non-Abelian one at $k_x = k_y = 0$. One also checks that when $\lambda = 1$, the growth rate of the unstable mode $\sqrt{b_- gB}$ is maximal and equal to $\sqrt{gB}$. When λ goes to zero or infinity, the growth rate tends to zero.

It follows from Eq. (79) that a behavior of the solution $f_-(\tau)$ depends of the sign of $\nu^2/\tau^2 - b_-^2$. For short times when $\nu^2/\tau^2 - b_-^2$ is positive, the function $f_-(\tau)$ oscillates around zero, and for later times when $\nu^2/\tau^2 - b_-^2$ is negative, the function grows exponentially, as discussed in [13,15,16]. When the field fluctuation is independent of space-time rapidity η, it is invariant under boosts along the z-axis and $\nu = 0$. Then, the function $f_-(\tau)$ starts exponentially growing right from $\tau = 0$. If the field fluctuation varies with η and $\nu > 0$, the exponential growth is delayed to $\tau = \nu/b_-$.

When the stability analysis of a uniform chromomagnetic field is performed using the Minkowski coordinates with time t not the proper time τ [21,22], plane waves are solutions of the linearized equations of motion, and the unstable modes start exponentially growing at $t = 0$. So, there is no delay of the growth, but the growth rate is reduced from $\sqrt{gB}$ to $\sqrt{gB - k_z^2}$, where k_z is the longitudinal momentum analogous to ν.

B. Effect of transverse momenta

Let us now discuss how finite momenta k_x or k_y influence the stability of the uniform magnetic field. If $k_x \neq 0$ or $k_y \neq 0$, Eq. (75) is a cubic one. Although the roots of such an equation are given by the Cardano formula, their structure is rather complex. So, for finite k_x or k_y we solve Eq. (75) perturbatively, assuming that $gB \gg k_x^2$ and $gB \gg k_y^2$. The solutions (77) are now the zeroth-order solution denoted as $\Lambda^{(0)}$, while the first-order solutions are assumed to be of the form

$$\Lambda^{(1)} = \Lambda^{(0)} + c_x k_x^2 + c_y k_y^2, \tag{83}$$

where the coefficients c_x and c_y are to be found. Substituting the formula (83) into Eq. (75), neglecting the terms that are quadratic and cubic in k_x^2 and k_y^2, and using the zeroth-order equations satisfied by $\Lambda^{(0)}$, one finds

$$\begin{aligned}
&-\big(c_x k_x^2 + c_y k_y^2\big)[(\Box + \lambda^{-2} gB - \Lambda^{(0)})(\Box + \lambda^2 gB - \Lambda^{(0)}) + (\Box + (\lambda^2 + \lambda^{-2}) gB - \Lambda^{(0)})(\Box + \lambda^2 gB - \Lambda^{(0)})\\
&\quad + (\Box + (\lambda^2 + \lambda^{-2}) gB - \Lambda^{(0)})(\Box + \lambda^{-2} gB - \Lambda^{(0)}) - 4g^2B^2] = 4\lambda^{-2} gB\, k_x^2 (\Box + \lambda^2 gB - \Lambda^{(0)})\\
&\quad + 4\lambda^2 gB\, k_y^2 (\Box + \lambda^{-2} gB - \Lambda^{(0)}).
\end{aligned} \tag{84}$$

Since Eq. (84) must be solved for any k_x^2 and any k_y^2, one finds the coefficients c_x and c_y putting $k_y^2 = 0$ and $k_x^2 = 0$, respectively. Thus, one obtains

$$c_x = -\frac{4\lambda^{-2}(\lambda^2 - \bar{\Lambda}^{(0)})}{(\lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^2 - \bar{\Lambda}^{(0)}) + (\lambda^2 + \lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^2 - \bar{\Lambda}^{(0)}) + (\lambda^2 + \lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^{-2} - \bar{\Lambda}^{(0)}) - 4}, \tag{85}$$

$$c_y = -\frac{4\lambda^{2}(\lambda^{-2} - \bar{\Lambda}^{(0)})}{(\lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^2 - \bar{\Lambda}^{(0)}) + (\lambda^2 + \lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^2 - \bar{\Lambda}^{(0)}) + (\lambda^2 + \lambda^{-2} - \bar{\Lambda}^{(0)})(\lambda^{-2} - \bar{\Lambda}^{(0)}) - 4}, \tag{86}$$

where

$$\bar{\Lambda}^{(0)} \equiv \frac{\Lambda^{(0)} - \Box}{gB}. \tag{87}$$

Using the zeroth-order solutions $\Lambda_1^{(0)}$ and $\Lambda_\pm^{(0)}$, we obtain

$$c_x^1 = = -\frac{4}{3}\lambda^{-4}, \quad c_y^1 = -\frac{4}{3}\lambda^4, \tag{88}$$

$$c_x^\pm = -\frac{4\lambda^{-2}(-\lambda^{-2} + \lambda^2 \mp \sqrt{(\lambda^{-2} + \lambda^2)^2 + 12})}{\lambda^4 + \lambda^{-4} \mp (\lambda^2 + \lambda^{-2})\sqrt{(\lambda^{-2} + \lambda^2)^2 + 12} + 14}, \tag{89}$$

$$c_y^\pm = -\frac{4\lambda^{2}(\lambda^{-2} - \lambda^2 \mp \sqrt{(\lambda^{-2} + \lambda^2)^2 + 12})}{\lambda^4 + \lambda^{-4} \mp (\lambda^2 + \lambda^{-2})\sqrt{(\lambda^{-2} + \lambda^2)^2 + 12} + 14}. \tag{90}$$

For $\lambda = 1$ the formulas (88), (89), and (90) simplify to

$$c_x^1 = c_y^1 = -\frac{4}{3}, \quad c_x^\pm = c_y^\pm = \pm\frac{2}{2 \mp 1}. \tag{91}$$

Knowing the approximate eigenvalues of the matrix from Eq. (72), we can write down the equations of motion as

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} + b_1^2 + \big(c_x^1 + 1\big)k_x^2 + \big(c_y^1 + 1\big)k_y^2\right) f_1(\tau) = 0, \tag{92}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} \pm b_\pm^2 + (c_x^\pm + 1)k_x^2 + (c_y^\pm + 1)k_y^2\right) f_\pm(\tau) = 0. \tag{93}$$

Since $k_x^2 \ll gB$ and $k_y^2 \ll gB$, a character of the solutions of Eqs. (92) and (93) is similar to those of Eqs. (78) and (79). The solutions f_1 and f_+ are stable while f_- can be unstable. So, we focus on the latter one, which is of our main interest. The solution is unstable if $\nu^2/\tau^2 - b_-^2 + (c_x^- + 1)k_x^2 + (c_y^- + 1)k_y^2$ is negative. Since $c_x^- + 1$ and $c_y^- + 1$ are both positive for any λ, one finds that finite momenta k_x and k_y tend to stabilize the solution. This is clearly seen in the case of $\lambda = 1$ when, as already discussed, the growth rate $\sqrt{b_- gB}$ is maximal when $k_x = k_y = 0$. For $\lambda = 1$, the equation of f_- reads

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \frac{\nu^2}{\tau^2} - gB + \frac{1}{3}k_T^2\right)f_-(\tau) = 0, \tag{94}$$

where $k_T^2 \equiv k_x^2 + k_y^2$. The growth rate of the unstable mode is $\sqrt{gB - k_T^2/3}$. However, we cannot conclude that the instability is absent if $k_T^2 \geqslant 3gB$, as k_T^2 is assumed to be small when compared to gB. We note that in the case of the Nielsen-Olesen mode in the Abelian configuration of a chromomagnetic field, the growth rate is not influenced by the transverse momentum, and it equals $\sqrt{gB}$. A comparison of unstable modes in Abelian and non-Abelian configurations of a uniform chromomagnetic field is presented in Fig. 5 of our earlier work [21], where the stability analysis is performed using the Minkowski coordinates with time not the proper time.

C. Special case: $a_a^\eta \neq 0$ & $a_a^x = a_a^y = 0$

Let us now discuss purely longitudinal dynamics. When $a_a^x = a_a^y = 0$, Eqs. (34)–(45) read

$$\partial_\eta\partial_\tau\left(\tau^2 a_1^\eta\right) = 0, \tag{95}$$
$$\partial_\eta\partial_\tau\left(\tau^2 a_2^\eta\right) = 0, \tag{96}$$
$$\partial_\eta\partial_\tau\left(\tau^2 a_3^\eta\right) = 0, \tag{97}$$
$$\partial_\eta\left(\partial_x a_1^\eta - g a_2^\eta \bar{A}_3^x\right) = 0, \tag{98}$$
$$\partial_\eta\left(\partial_x a_2^\eta + g a_1^\eta \bar{A}_3^x\right) = 0, \tag{99}$$
$$\partial_\eta\partial_x a_3^\eta = 0, \tag{100}$$
$$\partial_\eta\left(\partial_y a_1^\eta - g a_3^\eta \bar{A}_2^y\right) = 0, \tag{101}$$
$$\partial_\eta\partial_y a_2^\eta = 0, \tag{102}$$
$$\partial_\eta\left(\partial_y a_3^\eta + g a_1^\eta \bar{A}_2^y\right) = 0, \tag{103}$$
$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2 + g^2\left(\bar{A}_3^x\bar{A}_3^x + \bar{A}_2^y\bar{A}_2^y\right)\right)a_1^\eta - 2g\bar{A}_3^x\partial_x a_2^\eta + 2g\bar{A}_2^y\partial_y a_3^\eta = 0, \tag{104}$$
$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2 + g^2\bar{A}_3^x\bar{A}_3^x\right)a_2^\eta + 2g\bar{A}_3^x\partial_x a_1^\eta = 0, \tag{105}$$
$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2 + g^2\bar{A}_2^y\bar{A}_2^y\right)a_3^\eta - 2g\bar{A}_2^y\partial_y a_1^\eta = 0. \tag{106}$$

One sees that Eqs. (95)–(103) are solved if a_a^η is independent of η, which is the case assumed further on. Keeping in mind that $\bar{A}_2^y = \lambda\sqrt{B/g}$ and $\bar{A}_3^x = \lambda^{-1}\sqrt{B/g}$, Eqs. (104), (105), and (106) can be rewritten in matrix notation as

$$\begin{bmatrix} \Box + (\lambda^2 + \lambda^{-2})gB & -2i\lambda^{-1}\sqrt{gB}\,k_x & 2i\lambda\sqrt{gB}\,k_y \\ 2\lambda^{-1}i\sqrt{gB}\,k_x & \Box + \lambda^{-2}gB & 0 \\ -2\lambda i\sqrt{gB}\,k_y & 0 & \Box + \lambda^2 gB \end{bmatrix}\begin{bmatrix} a_1^\eta \\ a_2^\eta \\ a_3^\eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{107}$$

where we have assumed that the functions a_a^η depend on x and y through $e^{i(k_x x + k_y y)}$ and consequently

$$\Box \equiv \partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_x^2 + k_y^2. \tag{108}$$

When $k_x = k_y = 0$, Eq. (107) is diagonal and it provides three equations,

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + (\lambda^{-2} + \lambda^2)gB\right)a_1^\eta = 0, \tag{109}$$
$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \lambda^{-2}gB\right)a_2^\eta = 0, \tag{110}$$
$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + \lambda^2 gB\right)a_3^\eta = 0. \tag{111}$$

The solutions are $a_i^\eta \sim J_0(\sigma_i \tau)$ with

$$\sigma_1 = \sqrt{(\lambda^{-2} + \lambda^2)gB}, \quad \sigma_2 = \lambda^{-1}\sqrt{gB}, \quad \sigma_3 = \lambda\sqrt{gB}. \tag{112}$$

The solutions are stable.

When $k_x \neq 0$ or $k_y \neq 0$, Eq. (107) needs to be diagonalized. When $\lambda = 1$, one easily finds three equations,

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_T^2 + gB\right) f_1 = 0, \tag{113}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau + k_T^2 + \frac{1}{2}\big(3gB \pm \sqrt{g^2B^2 + 16gBk_T^2}\ \big)\right) f_\pm = 0. \tag{114}$$

Because $k_T^2 + gB > 0$ and

$$k_T^2 + \frac{1}{2}\big(3gB \pm \sqrt{g^2B^2 + 16gBk_T^2}\ \big) > 0,$$

the solutions are stable.

We have explicitly shown above that the uniform longitudinal chromomagnetic field is stable under purely longitudinal fluctuations in two special cases: (i) λ arbitrary and $k_x = k_y = 0$, (ii) $\lambda = 1$ and k_x, k_y are arbitrary. When λ, k_x, and k_y are arbitrary, the situation is more complicated, but the special cases suggest that the solutions of interest are stable.

V. STABILITY OF ELECTRIC FIELD

We consider here the stability of the uniform electric field E along the collision axis z. The field is in the Abelian configuration given by the formula (12). In Minkowski coordinates, the potential that generates the electric field $E_a^i = \delta^{iz}\delta^{a1}E$ and obeys the gauge condition $t\bar{A}^t - z\bar{A}^z = 0$ is

$$\bar{A}_a^\mu = -\tfrac{1}{2}\delta^{a1}(zE, 0, 0, tE). \tag{115}$$

In Milne coordinates, the potential is

$$\bar{A}_a^\mu = -\tfrac{1}{2}\delta^{a1}(0, 0, 0, E), \quad \bar{A}_\mu^a = \tfrac{1}{2}\delta^{a1}(0, 0, 0, \tau^2 E). \tag{116}$$

The problem of stability of a uniform chromoelectric field in the Abelian configuration was studied in Minkowski coordinates using the axial gauge $\bar{A}^z = a^z = 0$ in [10] and the Lorentz gauge $\partial_\mu \bar{A}^\mu = 0$ combined with the background gauge $D_\mu a^\mu = 0$, where the covariant derivative D_μ includes the background potential, in our earlier work [21]. The problem was also studied in the context of glasma [16], using the Milne coordinates and Fock-Schwinger gauge (4). In our analysis, which is presented below, we follow the study in Ref. [16], clarifying and improving some points.

With the background potential (116), the linearized Yang-Mills equations (25)–(28) split into color components are

$$-\partial_x\partial_\tau a_1^x - \partial_y\partial_\tau a_1^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_1^\eta\big) = 0, \tag{117}$$

$$-\partial_x\partial_\tau a_2^x - \partial_y\partial_\tau a_2^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_2^\eta\big) - g\tau^2\bar{A}_1^\eta\partial_\tau a_3^\eta = 0, \tag{118}$$

$$-\partial_x\partial_\tau a_3^x - \partial_y\partial_\tau a_3^y - \frac{1}{\tau^2}\,\partial_\eta\partial_\tau\big(\tau^2 a_3^\eta\big) + g\tau^2\bar{A}_1^\eta\partial_\tau a_2^\eta = 0, \tag{119}$$

$$\partial_\tau^2 a_1^x + \frac{1}{\tau}\,\partial_\tau a_1^x - \partial_y\big(\partial_y a_1^x - \partial_x a_1^y\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_1^x - \tau^2\partial_x a_1^\eta\big) = 0, \tag{120}$$

$$\partial_\tau^2 a_2^x + \frac{1}{\tau}\,\partial_\tau a_2^x - \partial_y\big(\partial_y a_2^x - \partial_x a_2^y\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_2^x - \tau^2\partial_x a_2^\eta + g\tau^2\bar{A}_1^\eta a_3^x\big) - g\bar{A}_1^\eta\big(\partial_\eta a_3^x - \tau^2\partial_x a_3^\eta\big) + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta a_2^x = 0, \tag{121}$$

$$\partial_\tau^2 a_3^x + \frac{1}{\tau}\,\partial_\tau a_3^x - \partial_y\big(\partial_y a_3^x - \partial_x a_3^y\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_3^x - \tau^2\partial_x a_3^\eta - g\tau^2\bar{A}_1^\eta a_2^x\big) + g\bar{A}_1^\eta\big(\partial_\eta a_2^x - \tau^2\partial_x a_2^\eta\big) + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta a_3^x = 0, \tag{122}$$

$$\partial_\tau^2 a_1^y + \frac{1}{\tau}\,\partial_\tau a_1^y - \partial_x\big(\partial_x a_1^y - \partial_y a_1^x\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_1^y - \tau^2\partial_y a_1^\eta\big) = 0, \tag{123}$$

$$\partial_\tau^2 a_2^y + \frac{1}{\tau}\,\partial_\tau a_2^y - \partial_x\big(\partial_x a_2^y - \partial_y a_2^x\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_2^y - \tau^2\partial_y a_2^\eta + g\tau^2\bar{A}_1^\eta a_3^y\big) - g\bar{A}_1^\eta\big(\partial_\eta a_3^y - \tau^2\partial_y a_3^\eta\big) + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta a_2^y = 0, \tag{124}$$

$$\partial_\tau^2 a_3^y + \frac{1}{\tau}\,\partial_\tau a_3^y - \partial_x\big(\partial_x a_3^y - \partial_y a_3^x\big) - \frac{1}{\tau^2}\,\partial_\eta\big(\partial_\eta a_3^y - \tau^2\partial_y a_3^\eta - g\tau^2\bar{A}_1^\eta a_2^y\big) + g\bar{A}_1^\eta\big(\partial_\eta a_2^y - \tau^2\partial_y a_2^\eta\big) + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta a_3^y = 0, \tag{125}$$

$$\partial_\tau^2\big(\tau^2 a_1^\eta\big) - \frac{1}{\tau}\partial_\tau(\tau^2 a_1^\eta) - \partial_x\big(\tau^2\partial_x a_1^\eta - \partial_\eta a_1^x\big) - \partial_y\big(\tau^2\partial_y a_1^\eta - \partial_\eta a_1^y\big) = 0, \tag{126}$$

$$\partial_\tau^2\big(\tau^2 a_2^\eta\big) - \frac{1}{\tau}\partial_\tau(\tau^2 a_2^\eta) - \partial_x\big(\tau^2\partial_x a_2^\eta - \partial_\eta a_2^x - g\tau^2 a_3^x\bar{A}_1^\eta\big) - \partial_y\big(\tau^2\partial_y a_2^\eta - \partial_\eta a_2^y - g\tau^2 a_3^y\bar{A}_1^\eta\big) = 0, \tag{127}$$

$$\partial_\tau^2\big(\tau^2 a_3^\eta\big) - \frac{1}{\tau}\partial_\tau(\tau^2 a_3^\eta) - \partial_x\big(\tau^2\partial_x a_3^\eta - \partial_\eta a_3^x + g\tau^2 a_2^x\bar{A}_1^\eta\big) - \partial_y\big(\tau^2\partial_y a_3^\eta - \partial_\eta a_3^y + g\tau^2 a_2^y\bar{A}_1^\eta\big) = 0. \tag{128}$$

As one observes, the fluctuating potential a_1^μ is decoupled not only from a_2^μ and a_3^μ but from the background field as well. So, a_1^μ describes free waves that are not discussed anymore. However, we still have a set of eight equations. Following [16], we consider a simplified situation when an evolution of longitudinal and transverse potential components is treated separately. Specifically, we discuss two special cases that allow one to reveal characteristic features of the problem.

A. Special case: $a_a^x = a_a^y = 0$ & $a_a^\eta \neq 0$

When $a_a^x = a_a^y = 0$ and $a_a^\eta \neq 0$, Eqs. (117)–(128) are

$$\left(\partial_\tau + \frac{2}{\tau}\right)\partial_\eta a_2^\eta + g\tau^2\bar{A}_1^\eta\partial_\tau a_3^\eta = 0, \tag{129}$$

$$\left(\partial_\tau + \frac{2}{\tau}\right)\partial_\eta a_3^\eta - g\tau^2\bar{A}_1^\eta\partial_\tau a_2^\eta = 0, \tag{130}$$

$$\partial_x\big(\partial_\eta a_2^\eta + g\tau^2\bar{A}_1^\eta a_3^\eta\big) = 0, \tag{131}$$

$$\partial_x\big(\partial_\eta a_3^\eta - g\tau^2\bar{A}_1^\eta a_2^\eta\big) = 0, \tag{132}$$

$$\partial_y\big(\partial_\eta a_2^\eta + g\tau^2\bar{A}_1^\eta a_3^\eta\big) = 0, \tag{133}$$

$$\partial_y\big(\partial_\eta a_3^\eta - g\tau^2\bar{A}_1^\eta a_2^\eta\big) = 0, \tag{134}$$

$$\left(\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2\right)a_2^\eta = 0, \tag{135}$$

$$\left(\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2\right)a_3^\eta = 0. \tag{136}$$

Introducing the functions

$$H^\pm(\tau,x,y,\eta) \equiv a_2^\eta(\tau,x,y,\eta) \pm i a_3^\eta(\tau,x,y,\eta), \tag{137}$$

Eqs. (129)–(136) are written as

$$\left[\left(\partial_\tau + \frac{2}{\tau}\right)\partial_\eta \mp ig\tau^2\bar{A}_1^\eta\partial_\tau\right]H^\pm = 0, \tag{138}$$

$$\partial_x\big(\partial_\eta \mp ig\tau^2\bar{A}_1^\eta\big)H^\pm = 0, \tag{139}$$

$$\partial_y\big(\partial_\eta \mp ig\tau^2\bar{A}_1^\eta\big)H^\pm = 0, \tag{140}$$

$$\left(\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_x^2 - \partial_y^2\right)H^\pm = 0. \tag{141}$$

If $\partial_x H^\pm \neq 0$ and $\partial_y H^\pm \neq 0$, Eqs. (139) and (140) are solved if the functions $H^\pm$ obey

$$\big(\partial_\eta \mp ig\tau^2\bar{A}_1^\eta\big)H^\pm = 0. \tag{142}$$

Equation (142) is solved by $H^\pm \sim \exp(\pm ig\tau^2\bar{A}_1^\eta\,\eta)$, which when substituted into Eq. (138) gives $H^\pm = 0$. So, we conclude that if $a_a^x = a_a^y = 0$, then $a_a^\eta = 0$ as well. In other words, the purely longitudinal dynamics is trivial.

B. Special case: $a_a^\eta = 0$ & $a_a^x \neq 0$, $a_a^y \neq 0$

When $a_a^\eta = 0$ and $a_a^x \neq 0$ or $a_a^y \neq 0$, Eqs. (117)–(128) read

$$\partial_\tau\Big(\partial_x a_2^x + \partial_y a_2^y\Big) = 0, \tag{143}$$

$$\partial_\tau\Big(\partial_x a_3^x + \partial_y a_3^y\Big) = 0, \tag{144}$$

$$\Big(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\Big)a_2^x + \partial_x\partial_y a_2^y - 2g\bar{A}_1^\eta\partial_\eta a_3^x = 0, \tag{145}$$

$$\Big(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\Big)a_3^x + \partial_x\partial_y a_3^y + 2g\bar{A}_1^\eta\partial_\eta a_2^x = 0, \tag{146}$$

$$\Big(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\Big)a_2^y + \partial_y\partial_x a_2^x - 2g\bar{A}_1^\eta\partial_\eta a_3^y = 0, \tag{147}$$

$$\Big(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\Big)a_3^y + \partial_y\partial_x a_3^x + 2g\bar{A}_1^\eta\partial_\eta a_2^y = 0, \tag{148}$$

$$\partial_\eta\big(\partial_x a_2^x + \partial_y a_2^y\big) + g\tau^2\bar{A}_1^\eta\big(\partial_x a_3^x + \partial_y a_3^y\big) = 0, \tag{149}$$

$$\partial_\eta\big(\partial_x a_3^x + \partial_y a_3^y\big) - g\tau^2\bar{A}_1^\eta\big(\partial_x a_2^x + \partial_y a_2^y\big) = 0. \tag{150}$$

Equations (143),(144) and (149),(150) are solved, respectively, if

$$\partial_x a_2^x = -\partial_y a_2^y, \qquad \partial_x a_3^x = -\partial_y a_3^y, \tag{151}$$

which substituted into Eqs. (145)–(148) give

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\right)a_2^x - 2g\bar{A}_1^\eta\partial_\eta a_3^x = 0, \tag{152}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\right)a_3^x + 2g\bar{A}_1^\eta\partial_\eta a_2^x = 0, \tag{153}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\right)a_2^y - 2g\bar{A}_1^\eta\partial_\eta a_3^y = 0, \tag{154}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 + g^2\tau^2\bar{A}_1^\eta\bar{A}_1^\eta\right)a_3^y + 2g\bar{A}_1^\eta\partial_\eta a_2^y = 0. \tag{155}$$

Introducing the functions

$$X^\pm \equiv a_2^x \pm i a_3^x, \qquad Y^\pm \equiv a_2^y \pm i a_3^y, \tag{156}$$

Eqs. (152)–(155) are diagonalized as

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 \mp igE\,\partial_\eta + \frac{1}{4}g^2\tau^2E^2\right)X^\pm = 0, \tag{157}$$

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau - \partial_x^2 - \partial_y^2 - \frac{1}{\tau^2}\,\partial_\eta^2 \mp igE\,\partial_\eta + \frac{1}{4}g^2\tau^2E^2\right)Y^\pm = 0, \tag{158}$$

where we put $\bar{A}_1^\eta = -\frac{1}{2}E$. Since the equations of $X^\pm$ and $Y^\pm$ are identical, further on we discuss only Eq. (157).

Assuming that the functions $X^\pm$ depend on x, y, and η through $e^{i(k_x x + k_y y + \nu\eta)}$, Eq. (157) becomes

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau + k_x^2 + k_y^2 + \frac{\nu^2}{\tau^2} \pm gE\nu + \frac{1}{4}g^2\tau^2E^2\right)X^\pm = 0, \tag{159}$$

which is rewritten as

$$\left(\partial_\tau^2 + \frac{1}{\tau}\,\partial_\tau + k_T^2 + \frac{1}{\tau^2}\left(\nu \pm \frac{1}{2}g\tau^2E\right)^2\right)X^\pm = 0, \tag{160}$$

where $k_T^2 = k_x^2 + k_y^2$. In the short time limit when $\tau^2 \ll 2\nu/(gE)$, we deal with the Bessel equation of imaginary order $i\nu$ which is solved by the oscillatory function $J_{i\nu}(k_T\tau)$. In the long τ limit when $\tau^2 \gg 1/(gE)$, $\tau^2 \gg 2\nu/(gE)$, and $\tau^2 \gg k_T^2/(g^2E^2)$, Eq. (160) becomes independent of ν and gets the form

$$\left(\partial_\tau^2 + \frac{1}{4}g^2E^2\tau^2\right)X^\pm = 0. \tag{161}$$

The solution is $X^\pm \sim \exp\left(\pm \frac{i}{4}gE\,\tau^2\right)$, which oscillates with the period decreasing to zero as $\tau \to \infty$. So, the solutions of Eq. (160), which actually represent waves running away along the z-axis to plus and minus infinity, are stable.

VI. TEMPORAL EVOLUTION OF GLASMA BACKGROUND FIELD

The glasma fields are not stationary but they evolve in time. Consequently, our stability analysis is reliable if the rate of change of the background field is significantly smaller than the growth rate of instability found in Sec. IV. To check the condition, we consider the evolution of glasma fields using the proper time expansion [27,28]. The potentials $\alpha(\tau, \mathbf{x}_\perp)$ and $\boldsymbol{\alpha}_\perp(\tau, \mathbf{x}_\perp)$ are expanded in the proper time τ as

$$\alpha(\tau, \mathbf{x}_\perp) = \alpha^{(0)}(\mathbf{x}_\perp) + \tau\alpha^{(1)}(\mathbf{x}_\perp) + \tau^2\alpha^{(2)}(\mathbf{x}_\perp) + \cdots, \tag{162}$$

$$\boldsymbol{\alpha}_\perp(\tau, \mathbf{x}_\perp) = \boldsymbol{\alpha}_\perp^{(0)}(\mathbf{x}_\perp) + \tau\boldsymbol{\alpha}_\perp^{(1)}(\mathbf{x}_\perp) + \tau^2\boldsymbol{\alpha}_\perp^{(2)}(\mathbf{x}_\perp) + \cdots. \tag{163}$$

The zeroth-order functions are given by the boundary conditions (5) and (6), that is, $\alpha_\perp^{(0)}(\mathbf{x}_\perp) = \alpha_\perp(0, \mathbf{x}_\perp)$ and $\alpha_\perp^{i(0)}(\mathbf{x}_\perp) = \alpha_\perp^i(0, \mathbf{x}_\perp)$, with $i, j = x, y$.

One shows that the coefficients multiplying odd powers of τ in the series (162) and (163) vanish [28], while the second-order coefficients in the fundamental representation are given as

$$\alpha^{(2)} = \tfrac{1}{8}[\mathcal{D}^j, [\mathcal{D}^j, \alpha^{(0)}]], \tag{164}$$

$$\alpha_\perp^{i(2)} = \tfrac{1}{4}\epsilon^{zij}[\mathcal{D}^j, B], \tag{165}$$

where all quantities are taken at the same point $\mathbf{x}_\perp$ and $\mathcal{D}^i \equiv \partial^i - ig\alpha_\perp^{i(0)}$. The coefficients (164) and (165) are expressed through the precollision potentials β_1^i, β_2^i in the following way:

$$\begin{aligned}\alpha^{(2)} = \frac{g}{16}\big(&-i\partial^j\partial^j\big[\beta_1^i, \beta_2^i\big] - g\partial^j\big[\beta_1^j, \big[\beta_1^i, \beta_2^i\big]\big]\\ &- g\partial^j\big[\beta_2^j, \big[\beta_1^i, \beta_2^i\big]\big]\big] - g\big[\beta_1^j + \beta_2^j, \partial^j\big[\beta_1^i, \beta_2^i\big]\big]\\ &+ ig^2\big[\beta_1^j + \beta_2^j, \big[\beta_1^j\big[\beta_1^i, \beta_2^i\big]\big]\\ &+ ig^2\big[\beta_1^j + \beta_2^j, \big[\beta_2^j\big[\beta_1^i, \beta_2^i\big]\big]\big),\end{aligned} \tag{166}$$

$$\alpha_\perp^{i(2)} = \frac{g}{4}\epsilon^{zij}\epsilon^{zkl}\big(i\partial^j\big[\beta_1^k, \beta_2^l\big] + g\big[\beta_1^j + \beta_2^j, \big[\beta_1^k, \beta_2^l\big]\big]\big). \tag{167}$$

When β_1^i and β_2^i are independent of $\mathbf{x}_\perp$, the second-order contributions to α and $\boldsymbol{\alpha}_\perp$ are

$$\begin{aligned}\alpha^{(2)} = \frac{ig^3}{16}\big(&\big[\beta_1^j + \beta_2^j, \big[\beta_1^j\big[\beta_1^i, \beta_2^i\big]\big]\\ &+ \big[\beta_1^j + \beta_2^j, \big[\beta_2^j\big[\beta_1^i, \beta_2^i\big]\big]\big),\end{aligned} \tag{168}$$

$$\alpha_\perp^{i(2)} = \frac{g^2}{4}\epsilon^{zij}\epsilon^{zkl}\big[\beta_1^j + \beta_2^j, \big[\beta_1^k, \beta_2^l\big]\big]. \tag{169}$$

Now, let us assume that the precollision potentials, which are purely transverse, are $\boldsymbol{\beta}_1 = (\beta_1, 0)$ and $\boldsymbol{\beta}_2 = (0, \beta_2)$. Then, $[\beta_1^i, \beta_2^i]$ vanishes and so does the initial electric field (9). The initial magnetic field (10) equals $B = ig[\beta_1, \beta_2]$. The second-order contributions are

$$\alpha^{(2)} = 0, \tag{170}$$

$$\alpha_\perp^{x(2)} = \frac{g^2}{4}\epsilon^{zxy}\epsilon^{zxy}[\beta_2, [\beta_1, \beta_2]] = \frac{g^2}{4}[\beta_2, [\beta_1, \beta_2]], \tag{171}$$

$$\alpha_\perp^{y(2)} = \frac{g^2}{4}\epsilon^{zyx}\epsilon^{zxy}[\beta_1, [\beta_1, \beta_2]] = -\frac{g^2}{4}[\beta_1, [\beta_1, \beta_2]]. \tag{172}$$

Going to the adjoint representation of the SU(2) group, one finds

$$\alpha_{\perp a}^{x(2)} = \frac{g^2}{4}\big(\beta_2^b\beta_1^b\beta_2^a - \beta_2^b\beta_1^a\beta_2^b\big), \tag{173}$$

$$\alpha_{\perp a}^{y(2)} = -\frac{g^2}{4}\big(\beta_1^b\beta_1^b\beta_2^a - \beta_1^b\beta_1^a\beta_2^b\big). \tag{174}$$

If $\beta_1^a = \lambda^{-1}\delta^{a3}\sqrt{B/g}$ and $\beta_2^a = \lambda\delta^{a2}\sqrt{B/g}$, the second-order contributions to $\boldsymbol{\alpha}_{\perp a}$ are

$$\alpha_{\perp a}^{x(2)} = -\tfrac{1}{4}\delta^{a3}\lambda g^{1/2}B^{3/2}, \tag{175}$$

$$\alpha_{\perp a}^{y(2)} = -\tfrac{1}{4}\delta^{a2}\lambda^{-1}g^{1/2}B^{3/2}. \tag{176}$$

Taking into account the zeroth- and second-order contributions in proper time expansion, the function α_a and the x and

y components of the function $\boldsymbol{\alpha}_{\perp a}$ are equal to

$$\begin{aligned} \alpha_a &= O(\tau^4), \\ \alpha^x_{\perp a} &= \delta^{a3}\lambda^{-1}\sqrt{B/g}\big(1 - \tfrac{1}{4}\lambda^2 gB\tau^2 + O(\tau^4)\big), \\ \alpha^y_{\perp a} &= \delta^{a2}\lambda\sqrt{B/g}\big(1 - \tfrac{1}{4}\lambda^{-2} gB\tau^2 + O(\tau^4)\big). \end{aligned} \tag{177}$$

One observes that the potential (177) generates the zeroth- and second-order longitudinal magnetic field and the first-order transverse electric field.

VII. DISCUSSION AND CONCLUSIONS

In our stability analysis, the electric and magnetic fields are assumed to be space-time uniform. However, the glasma fields generated at the earliest phase of ultrarelativistic heavy-ion collisions are not uniform, neither spatially nor temporally. So, one asks to what extent our results apply to the description of real glasma.

The correlators of glasma fields, discussed, e.g., in Sec. III D of [30], show that the fields are spatially uniform in the transverse plane at a scale L that is in between Q_s^{-1} and $\Lambda_{\rm QCD}^{-1}$, where $Q_s \approx 2$ GeV is the saturation scale and $\Lambda_{\rm QCD} \approx 0.2$ GeV is the QCD confinement scale at which color charges are neutralized. Assuming that the domain, where the field is uniform, is a square centered at $\mathbf{r}_\perp = \mathbf{0}$ and demanding that the real potentials a^x_a and a^y_a vanish at the edge of the square, the wave vectors k_x, k_y should be replaced as

$$(k_x, k_y) \longrightarrow (2l_x + 1, 2l_y + 1)\frac{\pi}{L}, \tag{178}$$

where l_x, l_y are integer numbers. Consequently, a spectrum of eigenmodes becomes discrete and the unstable mode found in Sec. IV can disappear if the minimal momentum π/L is sufficient to stabilize it. Using the growth rate of the instability estimated as $\sqrt{gB - \frac{1}{3}k_T^2}$, one finds that due to the replacement (178), the instability appears if

$$gB > \frac{2\pi^2}{3L^2}. \tag{179}$$

Since $g \approx 1$ and $B \approx Q_s^2 \approx 4$ GeV^2, the condition (179) is satisfied for $L^{-1} \approx \Lambda_{\rm QCD} \approx 0.2$ GeV. Taking into account that the generation of chromodynamic fields in heavy-ion collisions is a random process and consequently a magnitude of the field and a size of the domain, where the field is uniform, vary both in an individual collision and from collision to collision, we expect that the condition (179) is not always satisfied, but it often is and then the unstable mode occurs.

The initial glasma fields are not stationary, and as discussed in Sec. VI, the magnetic field changes with the characteristic rate $gB\tau$, where we set $\lambda = 1$. It is smaller than the instability growth rate estimated as $\sqrt{gB}$ when $\tau < (gB)^{-1/2}$ but it is bigger when $\tau > (gB)^{-1/2}$. This suggests that the initial magnetic field can be treated as stationary only for a very short time. However, one should take into account that our estimate of the field rate of change is obtained in the second order of the proper time expansion. The calculations using the proper time expansion, which are presented in, e.g., [30], show that there are usually alternating signs of successive terms in the proper time expansion, and consequently the temporal evolution is significantly slower than a second-order result suggests. Therefore, we expect that the initial magnetic field can be treated as stationary for a timescale $(gB)^{-1/2}$ or even longer.

Let us now confront our findings with results of the simulations of glasma evolution [3,4], which actually were the main motivation of our work. The simulations showed that the glasma is unstable, and the instability was identified with the Weibel mode [3,4]. We first note that the fastest unstable mode found in [3,4] grows like $e^{\sqrt{\tau}}$ while that of the glasma initial magnetic field grows like e^{τ}. The discrepancy can be removed taking into account that the longitudinal magnetic field decreases, as our formulas (177) show. However, we are not going to pursue this path as there are more important reasons not to interpret the results of the simulations [3,4] as due to the glasma initial field instability.

When the fastest growing mode found in [3,4] is fitted with $e^{\gamma\tau}$, the maximal growth rate is $\gamma \approx 0.002\,72\, g^2\mu$, where μ is the surface density of color charges of incoming nuclei.[1] Using the authors' estimate $g^2\mu \approx 20$ fm^{-1} for LHC, we get $\gamma \approx 0.05$ fm^{-1}. The maximal growth rate of the unstable mode of the glasma initial field is $\sqrt{gB} \approx Q_s$ and it occurs for $k_T = 0$ and $\lambda = 1$. Estimating the saturation scale as $Q_s = 2$ GeV, the maximal growth rate is $\sqrt{gB} \approx 10$ fm^{-1}, which is 200 times bigger than that from [3,4]. Although, the growth rate is smaller for $k_T > 0$ and/or $\lambda \neq 1$, it is hard to expect that the two very different growth rates describe the same physical phenomenon.

In the glasma simulations [3,4], the instability shows up only if the initial condition includes fluctuations that depend on the space-time rapidity η and consequently violate the boost invariance. This is actually the crucial argument to identify the instability as the Weibel mode, which requires a finite longitudinal momentum [7]. The unstable mode of the initial glasma field can occur at any ν including $\nu = 0$ which corresponds to the boost invariant configuration. While the growth rate is independent of ν, the modes start growing at $\tau = \nu/\sqrt{gB}$. The mode with $\nu = 0$, which is independent of η, starts with no delay and consequently it is dynamically most important.

The instabilities of initial glasma fields we have studied analytically here are presumably responsible for a rapid temporal evolution of glasma field correlators investigated in [29] using numerical simulations. The authors of Ref. [29] found that the correlator of chromomagnetic fields, which are initially uniform, changes with a characteristic time of the order Q_s^{-1}. However, a more detailed analysis is necessary to confirm the supposition.

We conclude our considerations as follows. The initial glasma field configuration is unstable if the fields are sufficiently uniform both spatially and temporally. Since the process of generation of chromodynamic fields in heavy-ion collisions is stochastic and the field's characteristics fluctuate, we expect that the condition of uniformity is often satisfied.

[1] We have taken into account the factor of 2 as the growth of the energy-momentum tensor, not the growth of the field, was obtained in [3,4].

The time of the instability development is of order 0.1 fm/c, which is much shorter than that of the instability found in the glasma simulations [3,4], which is of order 10 fm/c. The fastest unstable mode of the initial glasma field is boost invariant in contrast to the Weibel mode advocated in [3,4], which requires breaking of the boost invariance. So, we conclude that the instability found in the study [3,4] is not the instability of the glasma initial field. To observe the initial glasma field instability, if it indeed occurs, one needs a glasma simulation of high temporal resolution, much higher than that from [3,4].

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APPENDIX: BESSEL EQUATIONS

The Bessel equation is

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{\alpha^2}{x^2}\right)f(x)=0, \tag{A1}$$

and its two linearly independent solutions are the Bessel functions

$$J_\alpha(x)\equiv\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}, \tag{A2}$$

$$Y_\alpha(x)\equiv\frac{J_\alpha(x)\,\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}, \tag{A3}$$

which oscillate around zero for $x\in\mathbb{R}$. At $x=0$ the functions $J_\alpha(x)$ are finite but $Y_\alpha(x)$ diverge. For $x\gg|\alpha^2-\frac{1}{4}|$ the following approximation holds:

$$J_\alpha(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{2}\alpha-\frac{\pi}{4}\right)+O(x^{-3/2}). \tag{A4}$$

Changing the variable $x=it$, Eq. (A1) becomes the modified Bessel equation

$$\left(\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-1-\frac{\alpha^2}{t^2}\right)g(t)=0, \tag{A5}$$

where $g(t)=f(it)$ and the two linearly independent solutions are the modified Bessel functions

$$I_\alpha(t)\equiv i^{-\alpha}J_\alpha(it)=\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\alpha+1)}\left(\frac{t}{2}\right)^{2m+\alpha}, \tag{A6}$$

$$K_\alpha(t)\equiv\frac{\pi}{2}\frac{I_{-\alpha}(t)-I_\alpha(t)}{\sin(\alpha\pi)}. \tag{A7}$$

The functions $I_\alpha(t)$ are finite at $t=0$ and grow exponentially with t for $t\in\mathbb{R}$. The functions $K_\alpha(t)$ are infinite at $t=0$ and decay exponentially as t grows. For $t\gg|\alpha^2-\frac{1}{4}|$ we have the approximation

$$I_\alpha(t)=\frac{e^t}{\sqrt{\pi t}}(1+O(t^{-1})). \tag{A8}$$

Changing the variable $x=a\tau$ in Eq. (A1) and the variable $t=a\tau$ in Eq. (A5), the Bessel and modified Bessel equations read

$$\left(\frac{d^2}{d\tau^2}+\frac{1}{\tau}\frac{d}{d\tau}+a^2-\frac{\alpha^2}{\tau^2}\right)h(\tau)=0, \tag{A9}$$

$$\left(\frac{d^2}{d\tau^2}+\frac{1}{\tau}\frac{d}{d\tau}-a^2-\frac{\alpha^2}{\tau^2}\right)h(\tau)=0, \tag{A10}$$

where $h(\tau)\equiv f(a\tau)$ or $h(\tau)\equiv g(a\tau)$. The solutions are $J_\alpha(a\tau)$, $Y_\alpha(a\tau)$ and $I_\alpha(a\tau)$, $K_\alpha(a\tau)$, respectively.

[1] E. Iancu and R. Venugopalan, in *Quark-Gluon Plasma 3*, edited by R. C. Hwa and X.-N. Wang (World Scientific, Singapore, 2004).
[2] F. Gelis, Int. J. Mod. Phys. A **28**, 1330001 (2013).
[3] P. Romatschke and R. Venugopalan, Phys. Rev. Lett. **96**, 062302 (2006).
[4] P. Romatschke and R. Venugopalan, Phys. Rev. D **74**, 045011 (2006).
[5] E. S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).
[6] St. Mrówczyński, Phys. Lett. B **314**, 118 (1993).
[7] St. Mrówczyński, B. Schenke, and M. Strickland, Phys. Rep. **682**, 1 (2017).
[8] J. E. Mandula, Phys. Lett. B **67**, 175 (1977).
[9] J. E. Mandula, Phys. Rev. D **14**, 3497 (1976).
[10] S. J. Chang and N. Weiss, Phys. Rev. D **20**, 869 (1979).
[11] P. Sikivie, Phys. Rev. D **20**, 877 (1979).
[12] T. N. Tudron, Phys. Rev. D **22**, 2566 (1980).
[13] K. Fukushima, Phys. Rev. C **76**, 021902(R) (2007).
[14] A. Iwazaki, Phys. Rev. C **77**, 034907 (2008).
[15] A. Iwazaki, Prog. Theor. Phys. **121**, 809 (2009).
[16] H. Fujii and K. Itakura, Nucl. Phys. A **809**, 88 (2008).
[17] H. Fujii, K. Itakura, and A. Iwazaki, Nucl. Phys. A **828**, 178 (2009).
[18] N. K. Nielsen and P. Olesen, Nucl. Phys. B **144**, 376 (1978).
[19] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[20] L. S. Brown and W. I. Weisberger, Nucl. Phys. B **157**, 285 (1979); L. S. Brown, *ibid.* **172**, 544(E) (1980).
[21] S. Bazak and St. Mrówczyński, Phys. Rev. D **105**, 034023 (2022).
[22] S. Bazak and St. Mrówczyński, Phys. Rev. D **106**, 034031 (2022).
[23] J. Berges, S. Scheffler, S. Schlichting, and D. Sexty, Phys. Rev. D **85**, 034507 (2012).
[24] T. Vachaspati, Phys. Rev. D **105**, 105011 (2022).
[25] J. Pereira and T. Vachaspati, Phys. Rev. D **106**, 096019 (2022).
[26] A. Kovner, L. D. McLerran, and H. Weigert, Phys. Rev. D **52**, 3809 (1995).
[27] R. J. Fries, J. I. Kapusta, and Y. Li, arXiv:nucl-th/0604054.
[28] G. Chen, R. J. Fries, J. I. Kapusta, and Y. Li, Phys. Rev. C **92**, 064912 (2015).
[29] M. Ruggieri, J. H. Liu, L. Oliva, G. X. Peng, and V. Greco, Phys. Rev. D **97**, 076004 (2018).
[30] M. E. Carrington, A. Czajka, and St. Mrówczyński, Phys. Rev. C **105**, 064910 (2022).