Transport theory beyond binary collisions

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Using the Schwinger-Keldysh technique, we derive the transport equations for a system of quantum scalar fields. We first discuss the general structure of the equations and then their collision terms. Taking into account up to three-loop diagrams in ϕ^3 model and up to four-loop diagrams in ϕ^4 model, we obtain transport equations which include the contributions of multiparticle collisions and particle production processes, in addition to mean-field effects and binary interactions.

I. INTRODUCTION

Transport theory is a very convenient tool to study many-body nonequilibrium systems, both relativistic and nonrelativistic. The kinetic equations, which play a central role in the transport approach, usually assume that dissipation processes are governed by binary collisions. However, when the system of interest is very dense one expects that multiparticle interactions will play a significant role. Such interactions are known to control spectra of fluctuations and transport properties of dense gases and liquids [1]. Furthermore, in relativistic systems a characteristic particle's kinetic energy is usually comparable to the particle's mass, and processes leading to particle production become important for a system's dynamics.

It is therefore expected that multiparticle interactions and production processes will play an important role in the dynamics of a relativistic quark-gluon plasma of high energy density. The importance of gluon multiplication in the process of the plasma's equilibration has been repeatedly stressed, see, e.g., [2–7]. The scattering of three gluons into three gluons has been studied in the context of thermalization in [8]. It has also been shown within the scalar field theory that the transport coefficient of bulk viscosity, as given by the Kubo formula, strongly depends on particle number changing processes [9,10]. We conclude that a complete description of the quark-gluon plasma based on transport theory requires the derivation of the relativistic transport equation which includes the multiparticle collisions and particle production processes, in addition to the mean-field effects and binary interactions. The form of such an equation has been postulated by some authors, see, e.g., [6,8–10], but a systematic derivation from first principles is lacking even in the simplest scalar field theory.

The Schwinger-Keldysh [11,12] formulation of quantum field theory provides a framework for the derivation of the transport equation. Kadanoff and Baym [13] developed a technique for nonrelativistic quantum systems which has been further generalized to relativistic systems [14–24]. We mention here only the papers which go beyond the mean-field or Vlasov approximation and provide a more or less systematic analysis of the (binary) collision term.

In this paper, we consider self-interacting scalar fields with cubic and quartic interaction terms and study their transport equations beyond the binary collision approximation. There are basically two separate parts of our study. The first part is the derivation of the general structure of the transport equations. Such an analysis has been published previously by one of us [18,20,25] and it is presented here only to provide a framework for the second part of the paper. The main steps of the derivation are the following. We define the contour Green's function with the time arguments on the contour in the complex time plane. This function is a key element of the Schwinger-Keldysh approach. After discussing its properties and relevance for nonequilibrium systems, we write down the exact equations of motion, i.e., the Dyson-Schwinger equations. We perform a Wigner transformation and do a gradient expansion by assuming that the system has macroscopic quasihomogeneity. The resulting pair of Dyson-Schwinger equations is converted into the transport and mass-shell equations which are satisfied by the Wigner function. We define the distribution function of usual probabilistic interpretation, and we find the transport equations satisfied by this function. The transport equation is derived to lowest order in the gradient expansion, which corresponds to the Markovian limit in which memory terms are neglected.

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In the second part of our study, we perform a perturbative analysis of the self-energies which enter the transport equation. We show that the Vlasov or mean-field terms are dominated by the lowest order tadpole diagrams, while binary collisions emerge from two-loop contributions. Multiparticle interactions and production processes appear at three-loop level in the ϕ^3 model and at four-loops in the ϕ^4 model. There are obviously higher loop contributions to both mean-field and binary collision terms but these higher loop contributions will not change these effects qualitatively, and for this reason are not studied here. Instead we obtain leading order contributions to the collision term that result from $2 \leftrightarrow 3$ processes in ϕ^3 model, and $2 \leftrightarrow 4$ and $3 \leftrightarrow 3$ processes in ϕ^4 model. To extract the processes represented by the three- and four-loop graphs, we work with the Keldysh representation, using a technique developed previously by one of us [26,27].

Multiparticle and production processes have already been discussed in the Keldysh-Schwinger approach to transport theory but, to our best knowledge, only for non-relativistic systems [28–30]. In this case the analysis is quite different. In the nonrelativistic limit, interactions are instantaneous and internal lines representing the potential interaction cannot be cut. Consequently, the extraction of the scattering matrix elements in the small coupling limit is considerably simpler than in the relativistic case. However, it should be noted that much progress has been achieved in resuming multiloop diagrams in the nonrelativistic approach [29].

The problem of off-mass-shell transport has recently attracted a lot attention in the literature [31–36]. The collision term for off-mass-shell transport contains important contributions from one- and two-loop self-energy diagrams which correspond to $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ processes. These processes are kinematically forbidden for on-mass-shell particles, and for this reason are not studied here.

Throughout this article we use natural units where $\hbar = c = 1$. The signature of the metric tensor is $(+, -, -, -)$.

II. PRELIMINARIES

We consider a system of real scalar fields with Lagrangian density of the form

$$\mathcal{L}(x) = \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) - \frac{g}{n!} \phi^n(x), \quad (1)$$

where n equals 3 or 4. The renormalization counterterms are omitted here. The field satisfies the equation of motion

$$[\partial^2 + m^2]\phi(x) = -\frac{g}{(n-1)!} \phi^{n-1}(x). \quad (2)$$

The energy-momentum tensor is defined as

$$T^{\mu\nu}(x) = \partial^\mu \phi(x) \partial^\nu \phi(x) - g^{\mu\nu} \mathcal{L}(x).$$

Subtracting the total derivative

$$\frac{1}{4} \partial^\mu \partial^\nu \phi^2(x) - g^{\mu\nu} \frac{1}{4} \partial^\sigma \partial_\sigma \phi^2(x),$$

and considering the case of free fields, we obtain an expression for the energy-momentum tensor which has a form that is convenient for our purposes

$$T_0^{\mu\nu}(x) = -\frac{1}{4} \phi(x) \overleftrightarrow{\partial}^\mu \overleftrightarrow{\partial}^\nu \phi(x). \quad (3)$$

III. GREEN'S FUNCTIONS

We define the contour Green's function as

$$i\Delta(x, y) \overset{\text{def}}{=} \langle \tilde{T} \phi(\mathrm{x}) \phi(\mathrm{y}) \rangle, \quad (4)$$

where $\langle \ldots \rangle$ denotes the ensemble average at time t_0 (usually identified with $-\infty$) and $\tilde{T}$ is the time ordering operation along the directed contour shown in Fig. 1. The parameter $t_{\max}$ is shifted to $+\infty$ in calculations. The time arguments are complex with an infinitesimal positive or negative imaginary part which locates them on the upper or lower branch of the contour. The ordering operation is defined as

$$\tilde{T} \phi(x) \phi(y) \overset{\text{def}}{=} \Theta(\mathrm{x}_0, \mathrm{y}_0) \phi(\mathrm{x}) \phi(\mathrm{y}) + \Theta(\mathrm{y}_0, \mathrm{x}_0) \phi(\mathrm{y}) \phi(\mathrm{x}), \quad (5)$$

where $\Theta(x_0, y_0)$ equals 1 if x_0 succeeds y_0 on the contour, and 0 if x_0 precedes y_0.

If the field develops a finite expectation value, as is the case when the symmetry is spontaneously broken, the contribution $\langle \phi(x) \rangle \langle \phi(y) \rangle$ is subtracted from the right-hand side of the equation defining the Green's function and one looks at field fluctuations around the expectation values (see, e.g., [18,20]). Since $\langle \phi(x) \rangle$ is expected to vanish in the models defined by the Lagrangians (1) we neglect this contribution in the definition of the Green's function.

The contour propagator involves four functions, which can be thought of as corresponding to propagation along the top branch of the contour, from the top branch to the bottom branch, from the bottom branch to the top branch, and along the bottom branch. Below, we define four
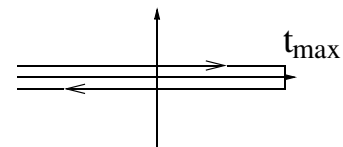

FIG. 1. The contour in the complex time plane for an evaluation of the operator expectation values.

Green's functions with real time arguments, and discuss their relationship to the contour Green's function. We define

$$i\Delta^>(x, y) \stackrel{\text{def}}{=} \langle \phi(x)\phi(y) \rangle, \qquad i\Delta^<(x, y) \stackrel{\text{def}}{=} \langle \phi(y)\phi(x) \rangle,$$

$$i\Delta^c(x, y) \stackrel{\text{def}}{=} \langle T^c\phi(x)\phi(y) \rangle, \qquad i\Delta^a(x, y) \stackrel{\text{def}}{=} \langle T^a\phi(x)\phi(y) \rangle, \quad (6)$$

where $T^c(T^a)$ indicates chronological (antichronological) time ordering

$$T^c\phi(x)\phi(y) \stackrel{\text{def}}{=} \Theta(x_0 - y_0)\phi(x)\phi(y)$$
$$+ \Theta(y_0 - x_0)\phi(y)\phi(x),$$

$$T^a\phi(x)\phi(y) \stackrel{\text{def}}{=} \Theta(y_0 - x_0)\phi(x)\phi(y)$$
$$+ \Theta(x_0 - y_0)\phi(y)\phi(x).$$

These functions are related to the contour Green's functions in the following manner:

$$\Delta^c(x, y) \equiv \Delta(x, y) \quad \text{for } x_0, y_0 \text{ on the upper branch,}$$
$$\Delta^a(x, y) \equiv \Delta(x, y) \quad \text{for } x_0, y_0 \text{ on the lower branch,}$$
$$\Delta^<(x, y) \equiv \Delta(x, y) \quad \text{for } x_0 \text{ on the upper branch, and } y_0 \text{ on the lower one,}$$
$$\Delta^>(x, y) \equiv \Delta(x, y) \quad \text{for } x_0 \text{ on the lower branch, and } y_0 \text{ on the upper one.} \quad (7)$$

The top branch of the contour is usually called the "1" branch and the bottom branch is called the "2" branch. Using this notation, the functions Δ^c, Δ^a, $\Delta^<$, and $\Delta^>$ are often written as

$$\Delta^c = \Delta_{11}, \qquad \Delta^a = \Delta_{22},$$
$$\Delta^< = \Delta_{12}, \qquad \Delta^> = \Delta_{21}. \quad (8)$$

The following identities can be obtained directly from the definitions given above:

$$\Delta^c_a(x, y) = \Theta(x_0 - y_0)\Delta^>_<(x, y) + \Theta(y_0 - x_0)\Delta^<_>(x, y),$$

$$(i\Delta(x, y))^\dagger = i\Delta^a_c(x, y) = i\Delta^a_c(y, x),$$

$$(i\Delta^>_<(x, y))^\dagger = i\Delta^>_<(x, y) = i\Delta^<_>(y, x), \quad (9)$$

where the $\dagger$ denotes Hermitian conjugation which involves an interchange of the arguments of the Green's function. Using Eqs. (5) and (8), it is easy to show that the four components of the contour Green's function are not independent but satisfy the relation:

$$\Delta^c(x, y) + \Delta^a(x, y) - \Delta^<(x, y) - \Delta^>(x, y) = 0. \quad (10)$$

In some situations, it is useful to work with retarded $(+)$, advanced $(-)$ and symmetric Green's functions. We make the usual definitions for these propagators:

$$i\Delta^+(x, y) \stackrel{\text{def}}{=} \Theta(x_0 - y_0)\langle[\phi(x), \phi(y)]\rangle$$
$$= i\Theta(x_0 - y_0)(\Delta^>(x, y) - \Delta^<(x, y)), \quad (11)$$

$$i\Delta^-(x, y) \stackrel{\text{def}}{=} -\Theta(y_0 - x_0)\langle[\phi(x), \phi(y)]\rangle$$
$$= i\Theta(y_0 - x_0)(\Delta^<(x, y) - \Delta^>(x, y)), \quad (12)$$

$$i\Delta^{\text{sym}}(x, y) \stackrel{\text{def}}{=} \langle\{\phi(x), \phi(y)\}\rangle = i(\Delta^>(x, y) + \Delta^<(x, y)), \quad (13)$$

where the curly brackets indicate an anticommutator.

It is easy to show that there is a simple relationship between the original set of propagators (Δ^c, Δ^a, $\Delta^<$ and $\Delta^>$) and the retarded, advanced and symmetric propagators (Δ^+, Δ^- and Δ^{sym}). In fact, since the four original propagators satisfy the constraint (9) and thus only three of them are independent, transferring between the two representations is equivalent to a change of basis. Henceforth, we will refer to the set of propagators Δ^+, Δ^-, and Δ^{sym} as the Keldysh representation of the contour ordered propagator, and the set Δ^c, Δ^a, $\Delta^<$, and $\Delta^>$ as the 1/2 representation. Using the definitions (5) and the relations (8), it is straightforward to connect the Keldysh to the 1/2 basis

$$\Delta^\pm(x, y) = \Delta^c(x, y) - \Delta^<_>(x, y),$$

$$\Delta^>_<(x, y) = \tfrac{1}{2}(\Delta^{\text{sym}}(x, y) \pm \Delta^+(x, y) \mp \Delta^-(x, y)),$$

$$\Delta^+(x, y) - \Delta^-(x, y) = \Delta^>(x, y) - \Delta^<(x, y). \quad (14)$$

The relations (9), (10), and (14) hold for real and complex fields. For pure real fields, which are studied here, there are extra relations among Green's functions. In particular, we will make use of the identity

$$\Delta^>(x, y) = \Delta^<(y, x). \quad (15)$$

We discuss briefly the physical interpretation of the Green's functions we have defined above. The functions Δ^c_a and $\Delta^\pm$ describe the propagation of a disturbance in which a single particle or antiparticle is added to the many-particle system at space-time point y and then is removed from it at a space-time point x. The function $\Delta^c(x, y)$ describes a particle disturbance propagating forward in time, and an antiparticle disturbance propagating backward in time. The meaning of $\Delta^a(x, y)$ is analogous but particles are propagated backward in time and antiparticles forward in time. In the zero density limit $\Delta^c(x, y)$ coincides with the Feynman propagator. In the case of the retarded (advanced) Green functions, both particles and antiparticles evolve forward (backward) in time.

The physical meaning of functions $\Delta^{\lessgtr}(x, y)$ is more transparent when one considers their Wigner transforms which are given by

$$\Delta^{\lessgtr}(X, p) \stackrel{\text{def}}{=} \int d^4u \, e^{ipu} \Delta^{\lessgtr}\left(X + \frac{1}{2}u, X - \frac{1}{2}u\right). \quad (16)$$

It is easy to show that the free-field energy-momentum tensor (3) averaged over the ensemble can be expressed as

$$\langle T_0^{\mu\nu}(X)\rangle = \int \frac{d^4p}{(2\pi)^4} p^\mu p^\nu i\Delta^{<}(X, p). \quad (17)$$

One recognizes the standard form of the energy-momentum tensor in kinetic theory with the function $i\Delta^{<}(X, p)$ playing the role of the density of particles with four-momentum p at the space-time point X. This observation leads us to consider $i\Delta^{<}(X, p)$ as the quantum analog of the classical distribution function. The function $i\Delta^{<}(X, p)$ is Hermitian but it is not positive definite, and thus the probabilistic interpretation is only approximately valid. One should also note that, in contrast to the classical distribution function, the function $i\Delta^{<}(X, p)$ can be non-zero for the off-mass-shell four-momenta. As will be explicitly shown in Sec. IX, one can extract the usual distribution function from $i\Delta^{<}(X, p)$ under specific conditions.

$$\delta^{(4)}(x, y) = \begin{cases} \delta^{(4)}(x - y) & \text{for} \quad x_0, y_0 \quad \text{from the upper branch,} \\ 0 & \text{for} \quad x_0, y_0 \quad \text{from the different branches,} \\ -\delta^{(4)}(x - y) & \text{for} \quad x_0, y_0 \quad \text{from the lower branch.} \end{cases}$$

To obtain equations for $\Delta^{\lessgtr}$ we split the self-energy into three parts as

$$\Pi(x, y) = \Pi_\delta(x)\delta^{(4)}(x, y) + \Pi^{>}(x, y)\Theta(x_0, y_0) + \Pi^{<}(x, y)\Theta(y_0, x_0), \quad (21)$$

where Π_δ corresponds to the tadpole contribution to the self-energy. With the help of the retarded and advanced Green's functions (11) and (12) and similar expressions for the retarded and advanced self-energies, we can rewrite Eqs. (19) and (20) to obtain

$$[\partial_x^2 + m^2 - \Pi_\delta(x)]\Delta^{\lessgtr}(x, y) = \int d^4x'[\Pi^{\lessgtr}(x, x')\Delta^{-}(x', y) + \Pi^{+}(x, x')\Delta^{\lessgtr}(x', y)], \quad (22)$$

$$[\partial_y^2 + m^2 - \Pi_\delta(y)]\Delta^{\lessgtr}(x, y) = \int d^4x'[\Delta^{\lessgtr}(x, x')\Pi^{-}(x', y) + \Delta^{+}(x, x')\Pi^{\lessgtr}(x', y)], \quad (23)$$

where all time integrations run from $-\infty$ to $+\infty$.

IV. EXACT EQUATIONS OF MOTION

Since the Green's function $i\Delta^{\lessgtr}(X, p)$ will be used to obtain the distribution function, we derive an equation which governs the evolution of $i\Delta^{\lessgtr}(X, p)$. We start with the Dyson-Schwinger equation for the *contour* Green's function which has the form

$$\Delta(x, y) = \Delta_0(x, y)$$
$$- \int_C d^4z \int_C d^4z' \Delta_0(x, z)\Pi(z, z')\Delta(z', y), \quad (18)$$

where $\Delta_0(x, y)$ is the free Green's function. We note that the integrals over z_0 and z_0' in the last term of Eq. (18) are performed along the contour.

The Dyson-Schwinger Eq. (18) can be rewritten as two equations:

$$[\partial_x^2 + m^2]\Delta(x, y) = -\delta^{(4)}(x, y)$$
$$+ \int_C d^4x' \Pi(x, x')\Delta(x', y), \quad (19)$$

$$[\partial_y^2 + m^2]\Delta(x, y) = -\delta^{(4)}(x, y)$$
$$+ \int_C d^4x' \Delta(x, x')\Pi(x', y), \quad (20)$$

where the function $\delta^{(4)}(x, y)$ is defined on the contour as

Similarly, the Dyson-Schwinger Eqs. (19) and (20) provide the equations satisfied by the retarded and advanced functions $\Delta^{\pm}(x, y)$ as

$$[\partial_x^2 + m^2 - \Pi_\delta(x)]\Delta^{\pm}(x, y) = -\delta^{(4)}(x - y)$$
$$+ \int d^4x' \Pi^{\pm}(x, x')\Delta^{\pm}(x', y), \quad (24)$$

$$[\partial_y^2 + m^2 - \Pi_\delta(y)]\Delta^{\pm}(x, y) = -\delta^{(4)}(x - y)$$
$$+ \int d^4x' \Delta^{\pm}(x, x')\Pi^{\pm}(x', y). \quad (25)$$

We note that the equations for the retarded and for advanced functions are decoupled from each other—there is no mixing of the retarded, advanced and symmetric components of the propagators.

V. APPROXIMATIONS

It is a very difficult task to study systems which are strongly inhomogeneous or strongly interacting—there are no general methods applicable to such systems. Thus, we assume that systems of interest are weakly inhomogeneous and weakly interacting. We discuss below the mathemati-

cal formulation of the corresponding approximations.

(1) The system is assumed to be weakly inhomogeneous in comparison with two different length scales.

    (a) We assume that the inhomogeneity length is large compared to the inverse of the characteristic momentum, i.e.,

$$|F(X, p)| \gg \left| \frac{\partial^2}{\partial X_\mu \partial p_\nu} F(X, p) \right|, \quad (26)$$

    where F is either the propagator or the self-energy. This assumption allows one to perform the gradient expansion.

    (b) The inhomogeneity length of the system is assumed to be large compared to the inverse characteristic mass of free quasiparticles (or the Comptom wavelength):

$$|F(X, p)| \gg \frac{1}{m^2} \left| \frac{\partial^2}{\partial X_\mu \partial X^\mu} F(X, p) \right|. \quad (27)$$

This condition justifies the so-called quasiparticle approximation. When the bare fields are massless or the free mass is much smaller than the dynamically generated effective mass m_*, the mass in Eq. (24) should be replaced by m_* [25].

(2) The assumption that the system is weakly interacting means that the coupling constant g in Eq. (1) is small and that all self-energies can be expanded perturbatively in g. The conditions of weak inhomogeneity and weak interaction will be used in five separate ways:

    (i) We will convert the Eqs. (22) and (23) into transport equations by performing Wigner transformations (16) on all Green's functions and self-energies. Using Eq. (26), we obtain the following set of translation rules:

$$\int d^4x' f(x, x')g(x', y) \longrightarrow f(X, p)g(X, p) + \frac{i}{2}\{f(X, p), g(X, p)\}, \qquad h(x)g(x, y) \longrightarrow h(X)g(X, p) - \frac{i}{2} \frac{\partial h(X)}{\partial X^\mu} \frac{\partial g(X, p)}{\partial p_\mu},$$

$$h(y)g(x, y) \longrightarrow h(X)g(X, p) + \frac{i}{2} \frac{\partial h(X)}{\partial X^\mu} \frac{\partial g(X, p)}{\partial p_\mu}, \qquad \partial_x^\mu f(x, y) \longrightarrow \left(-ip^\mu + \frac{1}{2} \partial^\mu \right) f(X, p), \qquad (28)$$

$$\partial_y^\mu f(x, y) \longrightarrow \left(ip^\mu + \frac{1}{2} \partial^\mu \right) f(X, p),$$

where we have introduced the Poisson-like bracket defined as

$$\{C(X, p), D(X, p)\} \equiv \frac{\partial C(X, p)}{\partial p_\mu} \frac{\partial D(X, p)}{\partial X^\mu}$$
$$- \frac{\partial C(X, p)}{\partial X^\mu} \frac{\partial D(X, p)}{\partial p_\mu}.$$

The function $h(x)$ is weakly dependent on x and we use the notation $\partial^\mu \equiv \frac{\partial}{\partial X_\mu}$. This calculation is done in Sec. VI.

(ii) The condition (27) allows one to drop the terms containing ∂^2. These terms and those involving $\mathrm{Im}\Pi^+$, which are discussed in (iv), are responsible for off-mass-shell contributions to the Green's functions $\Delta^{\gtrless}$.

(iii) The assumption (26) is used together with the condition of weak interaction to show that the gradient terms in the right-hand side of the transport Eq. (35) and mass-shell equation (36) can be dropped. This point is discussed in detail in Sec. VI.

(iv) The weakly interacting approximation is used in Sec. X. For $1/g \gg 1$ and a finite bare mass, we have

$$m^2 - \Pi_\delta - \mathrm{Re}\Pi^+ \gg |\mathrm{Im}\Pi^+|. \quad (29)$$

This assumption combined with the condition (27) allows us to obtain an expression for the spectral function that is proportional to $\delta(p^2 - m_*^2)$. We note that both of Eqs. (27) and (29) are commonly called "the quasiparticle approximation." This terminology is a consequence of the fact that there are two independent means by which the quasiparticle picture can be destroyed: strong inhomogeneity and strong interactions.

(v) The weakly interacting condition is also used in Sec. XII to perform a perturbative expansion of the self-energies in the collision term of the transport equation. Each term in this expansion represents a different physical contribution to the transport equation.

VI. EQUATIONS FOR $\Delta^{\gtrless}$

Applying the translation rules (28) to Eqs. (22) and (23), neglecting the terms proportional to ∂^2 due to the quasiparticle approximation (27), and taking the difference and sum of the equations, we obtain

$$\left[p^\mu \partial_\mu - \frac{1}{2}\partial_\mu \Pi_\delta(X)\partial_P^\mu\right]\Delta^{\gtrless}(X,p) = \frac{i}{2}(\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p))$$

$$- \frac{1}{4}\{\Pi^{\gtrless}(X,p), \Delta^+(X,p) + \Delta^-(X,p)\}$$

$$- \frac{1}{4}\{\Pi^+(X,p) + \Pi^-(X,p), \Delta^{\gtrless}(X,p)\}, \qquad (30)$$

$$[-p^2 + m^2 - \Pi_\delta(X)]\Delta^{\gtrless}(X,p) = \frac{1}{2}[\Pi^{\gtrless}(X,p)(\Delta^+(X,p) + \Delta^-(X,p)) + (\Pi^+(X,p) + \Pi^-(X,p))\Delta^{\gtrless}(X,p)]$$

$$+ \frac{i}{4}\{\Pi^>(X,p), \Delta^<(X,p)\} - \frac{i}{4}\{\Pi^<(X,p), \Delta^>(X,p)\}, \qquad (31)$$

where we have used the identity (14) applied to the Green's functions and self-energies. One recognizes Eq. (30) as a transport equation while Eq. (31) is called a mass-shell equation. We note that in the limit of free fields, Eqs. (30) and (31) become

$$p^\mu \partial_\mu \Delta_0^{\gtrless}(X,p) = 0, \qquad (32)$$

$$[p^2 - m^2]\Delta_0^{\gtrless}(X,p) = 0. \qquad (33)$$

Because of Eq. (33), $\Delta_0^{\gtrless}(X,p)$ is proportional to $\delta(p^2 - m^2)$, and consequently free quasiparticles are always on-mass-shell. We note that if the quasiparticle approximation (27) were *not* used, the mass-shell equation would have the form

$$\left[\frac{1}{4}\partial^2 - p^2 + m^2\right]\Delta_0^{\gtrless}(X,p) = 0, \qquad (34)$$

and the off-shell contribution to the Green's function $\Delta_0^{\gtrless}$ would be nonzero.

The Eqs. (30) and (31) can be rewritten in a more compact way:

$$\{p^2 - m^2 + \Pi_\delta(X) + \text{Re}\Pi^+(X,p), \Delta^{\gtrless}(X,p)\}$$
$$= i(\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p))$$
$$- \{\Pi^{\gtrless}(X,p), \text{Re}\Delta^+(X,p)\}, \qquad (35)$$

$$[p^2 - m^2 + \Pi_\delta(X) + \text{Re}\Pi^+(X,p)]\Delta^{\gtrless}(X,p)$$
$$= -\Pi^{\gtrless}(X,p)\text{Re}\Delta^+(X,p) - \frac{i}{4}\{\Pi^>(X,p), \Delta^<(X,p)\}$$

$$+ \frac{i}{4}\{\Pi^<(X,p), \Delta^>(X,p)\}, \qquad (36)$$
where

$$\text{Re}\Delta^\pm(X,p) \equiv \frac{1}{2}(\Delta^+(X,p) + \Delta^-(X,p)),$$
$$\text{Im}\Delta^\pm(X,p) \equiv \pm\frac{1}{2i}(\Delta^+(X,p) - \Delta^-(X,p)). \qquad (37)$$

We note that $(i\Delta^\pm(X,p))^\dagger = -i\Delta^\mp(X,p)$ which is a direct consequence of the definitions (11) and (12). The equation analogous to Eq. (35) was earlier derived in [13,14,25].

The gradient terms in the right-hand sides of Eqs. (35) and (36) are usually neglected. The justification for dropping these terms is considered below. The small parameter that characterizes the gradient expansion will be denoted ϵ. In Sec. XII we will calculate the self-energies using a perturbative expansion in the coupling constant g. For

the purposes of the discussion below, we will use a new parameter to characterize the coupling constant expansion. This new parameter is defined by $\lambda = g$ for ϕ^3 and $\lambda = \sqrt{g}$ for ϕ^4 theory. The parameter λ is introduced to simplify the notation: the leading order contributions to $\Pi_\delta(X) + \text{Re}\Pi^+(X,p)$ and $\Pi^{\gtrless}(X,p)$ are of order λ^2 and λ^4 respectively, in both ϕ^3 and ϕ^4 theory.[1]

Now, we compute the order of each term in Eq. (35). We can split the Poisson-like bracket on the left-hand side into two pieces. The first piece does not contain the self-energy and is of order ϵ. The second piece is of order $\lambda^2\epsilon$. The first term on the right-hand side (the term without the Poisson-like bracket) is of order λ^4, and the term with the Poisson-bracket (the gradient term) is of order $\lambda^4\epsilon$. Thus, we find that the gradient term on the right-hand side of Eq. (35) is of higher order than the remaining terms and can be dropped. Similarly, the gradient term on the right-hand side of Eq. (36) is of order $\lambda^4\epsilon$ and it is of higher order than the remaining terms.

We note that the argument presented above breaks down close to-equilibrium. At equilibrium the nongradient term on the right-hand side of Eq. (35) is identically zero, and thus cannot be considered to be bigger than the gradient term. In the equilibrium case, it has been shown that the gradient term does not need to be dropped but can be combined with the interaction term on the left-hand side to produce the usual Vlasov term [25]. The very-close-to-equilibrium situation is not fully understood.

VII. EQUATIONS FOR $\Delta^\pm$

We also write down the transport and mass-shell equations satisfied by the retarded and advanced Green's functions. Starting with Eqs. (24) and (25), one finds

$$\{p^2 - m^2 + \Pi_\delta(X) + \Pi^\pm(X,p), \Delta^\pm(X,p)\} = 0, \qquad (38)$$

$$[p^2 - m^2 + \Pi_\delta(X) + \Pi^\pm(X,p)]\Delta^\pm(X,p) = 1. \qquad (39)$$

---

[1]The different treatment of the ϕ^3 and ϕ^3 theory comes from the way the coupling constant enters the Lagrangian (1). If we had defined interaction terms of the form $g^2\phi^4/4!$ and $g\phi^3/3!$ we would avoid using the parameter λ.

We observe that the gradient terms drop out entirely in Eq. (39). The leading order corrections to this equation are second order in the gradient expansion. Equation (39) can be immediately solved to give

$$\Delta^{\pm}(X, p) = \frac{1}{p^2 - m^2 + \Pi_\delta(X) + \Pi^{\pm}(X, p)}. \qquad (40)$$

We note that any function $f(K)$ satisfies the equation $\{K, f(K)\} = 0$, for K an arbitrary function of X and p. Thus, the expression (40) solves Eq. (38) as well as Eq. (39).

The real and imaginary parts of $\Delta^{\pm}$, which we will need later, are

$$\mathrm{Re}\,\Delta^{\pm}(X, p) = \frac{p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p)}{(p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p))^2 + (\mathrm{Im}\Pi^+(X, p))^2}, \qquad (41)$$

$$\mathrm{Im}\,\Delta^{\pm}(X, p) = \frac{\mp\mathrm{Im}\Pi^+(X, p)}{(p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p))^2 + (\mathrm{Im}\Pi^+(X, p))^2}. \qquad (42)$$

Using the relations (11) and (12), we can obtain the Wigner transforms of the retarded and advanced functions for free fields

$$\Delta_0^{\pm}(X, p) = \frac{1}{p^2 - m^2 \pm ip_0 0^+}. \qquad (43)$$

Comparing the expressions (40) and (43), we find that in the limit $g \to 0$ the self-energies must satisfy

$$\mathrm{Im}\Pi^+(X, p) = -\mathrm{Im}\Pi^-(X, p) = \begin{cases} 0^+ & \text{for} \quad p_0 > 0, \\ 0^- & \text{for} \quad p_0 < 0. \end{cases} \qquad (44)$$

VII. SPECTRAL FUNCTION

In this section, we introduce another function which will be useful later on. The spectral function A is defined as

$$A(x, y) \overset{\text{def}}{=} \langle [\phi(x), \phi(y)] \rangle = i\Delta^>(x, y) - i\Delta^<(x, y)$$
$$= i\Delta^+(x, y) - i\Delta^-(x, y) = \mp 2\mathrm{Im}\Delta^{\pm}(x, y), \qquad (45)$$

where $[\phi(x), \phi(y)]$ denotes the field commutator. For real fields, which are studied here, $A(X, p) = -A(X, -p)$.

From the transport and mass-shell equations (35) and (36), one immediately finds equations for $A(X, p)$ of the form

$$\{p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p), A(X, p)\}$$
$$= 2\{\mathrm{Im}\Pi^+(X, p), \mathrm{Re}\Delta^+(X, p)\}, \qquad (46)$$

$$[p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p)]A(X, p)$$
$$= 2\mathrm{Im}\Pi^+(X, p)\mathrm{Re}\Delta^+(X, p). \qquad (47)$$

Substituting the formula (41) into the algebraic Eq. (47), we find the solution

$$A(X, p) = \frac{2\mathrm{Im}\Pi^+(X, p)}{(p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p))^2 + (\mathrm{Im}\Pi^+(X, p))^2}. \qquad (48)$$

It is easy to show that the function (48) solves Eq. (46) as well. In fact, the spectral function (48) can be found directly from Eq. (42) due to the last equality in Eq. (45).

The free spectral function can be obtained from the free retarded and advanced functions (43) by means of the relation (45). It can be also found from Eq. (48) using the condition (44). In either case one needs to make use of the well-known identity

$$\frac{1}{x \pm i0^+} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x). \qquad (49)$$

We obtain

$$A_0(X, p) = 2\pi\delta(p^2 - m^2)(\Theta(p_0) - \Theta(-p_0)). \qquad (50)$$

When $\mathrm{Im}\Pi^+ = 0$ the spectral function describes infinitely narrow quasiparticles with energies uniquely determined by their momenta. When $\mathrm{Im}\Pi^+ \neq 0$ the quasiparticles are of finite life time, and the spectral function gives the energy distribution of a quasiparticle with a given momentum.

IX. DISTRIBUTION FUNCTION

The distribution function $f(X, p)$ is defined through the equation

$$\Theta(p_0)A(X, p)f(X, p) \overset{\text{def}}{=} \Theta(p_0)i\Delta^<(X, p), \qquad (51)$$

where $A(X, p)$ is the spectral function (45). This definition is motivated by Eq. (17) which leads us to identify $\Delta^>$ as the quantum analogue of the classical distribution function. We note that $f(X, p)$ depends not on the three-vector **p** but on the four-vector p. Using Eq. (45) and the identity (15) in the form $\Delta^<(X, p) = \Delta^>(X, -p)$, we have

$$i\Delta^>(X, p) = \Theta(p_0)A(X, p)(f(X, p) + 1)$$
$$- \Theta(-p_0)A(X, p)f(X, -p), \qquad (52)$$

$$i\Delta^<(X, p) = \Theta(p_0)A(X, p)f(X, p)$$
$$- \Theta(-p_0)A(X, p)(f(X, -p) + 1). \qquad (53)$$

In the case of free fields, Eqs. (52) and (53) simplify to

$$i\Delta_0^>(X, p) = 2\pi\delta(p^2 - m^2)[\Theta(p_0)(f_0(X, \mathbf{p}) + 1)$$
$$+ \Theta(-p_0)f_0(X, -\mathbf{p})]$$
$$= \frac{\pi}{E_p}\delta(E_p - p_0)(f_0(X, \mathbf{p}) + 1)$$
$$+ \frac{\pi}{E_p}\delta(E_p + p_0)f_0(X, -\mathbf{p}), \qquad (54)$$

$$i\Delta_0^<(X, p) = 2\pi\delta(p^2 - m^2)[\Theta(p_0)f_0(X, \mathbf{p})$$
$$+ \Theta(-p_0)(f_0(X, -\mathbf{p}) + 1)]$$
$$= \frac{\pi}{E_p}\delta(E_p - p_0)f_0(X, \mathbf{p})$$
$$+ \frac{\pi}{E_p}\delta(E_p + p_0)(f_0(X, -\mathbf{p}) + 1), \qquad (55)$$

where the explicit form of the free spectral function (50) was used.

For future use, we write down the results for the propagators Δ_0^c, Δ_0^a and Δ_0^{sym} in terms of the free distribution function f_0. Using the free retarded and advanced functions (43) and the identity (49), we obtain

$$i\Delta_0^c(X, p) = i\Delta_0^+(X, p) + i\Delta_0^<(X, p),$$
$$= \frac{i}{p^2 - m^2 + i0^+} + 2\pi\delta(p^2 - m^2)$$
$$\times [\Theta(p_0)f_0(X, \mathbf{p}) + \Theta(-p_0)f_0(X, -\mathbf{p})] \quad (56)$$

$$i\Delta_0^a(X, p) = -i\Delta_0^-(X, p) + i\Delta_0^<(X, p),$$
$$= \frac{-i}{p^2 - m^2 - i0^+} + 2\pi\delta(p^2 - m^2)$$
$$\times [\Theta(p_0)f_0(X, \mathbf{p}) + \Theta(-p_0)f_0(X, -\mathbf{p})],$$
$$(57)$$

$$i\Delta_0^{\text{sym}}(X, p) = i\Delta_0^>(X, p) + i\Delta_0^<(X, p),$$
$$= 2\pi\delta(p^2 - m^2)[2\Theta(p_0)f_0(X, \mathbf{p})$$
$$+ 2\Theta(-p_0)f_0(X, -\mathbf{p}) + 1]. \qquad (58)$$

We note that in the case of equilibrium (homogeneous) systems, the familiar definition of the distribution function is

$$f^{\text{eq}}(\mathbf{p}) \overset{\text{def}}{=} \frac{1}{V}\langle a^\dagger(\mathbf{p})a(\mathbf{p})\rangle, \qquad (59)$$

where V denotes the system volume and $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are annihilation and creation operators, respectively. The definition (51) should reduce to (59) when an equilibrium (homogeneous) system is considered. We discuss this point below for the case of a noninteracting system for which the field that solves the equation of motion (2) can be written as

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}}(e^{-ikx}a(\mathbf{k}) + e^{ikx}a^\dagger(\mathbf{k})), \qquad (60)$$

where $k \equiv (E_k, \mathbf{k})$. Substituting the solution (60) into the definition (6) one finds

$$i\Delta_0^>(X, p) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}} \frac{d^3q}{\sqrt{(2\pi)^3 2E_q}}(2\pi)^4$$
$$\times \Bigg[e^{-i(k+q)X}\delta^{(4)}\Big(p - \frac{k - q}{2}\Big)\langle a(\mathbf{k})a(\mathbf{q})\rangle$$
$$+ e^{-i(k-q)X}\delta^{(4)}\Big(p - \frac{k + q}{2}\Big)\langle a(\mathbf{k})a^\dagger(\mathbf{q})\rangle$$
$$+ e^{i(k-q)X}\delta^{(4)}\Big(p + \frac{k + q}{2}\Big)\langle a^\dagger(\mathbf{k})a(\mathbf{q})\rangle$$
$$+ e^{i(k+q)X}\delta^{(4)}\Big(p + \frac{k - q}{2}\Big)\langle a^\dagger(\mathbf{k})a^\dagger(\mathbf{q})\rangle \Bigg].$$
$$(61)$$

Since we restrict ourselves to the consideration of homogeneous systems, we require that the function $\Delta_0^>(X, p)$ is independent of X. One ansatz that satisfies this constraint is given by

$$\langle a(\mathbf{k})a(\mathbf{q})\rangle = 0, \qquad \langle a^\dagger(\mathbf{k})a^\dagger(\mathbf{q})\rangle = 0, \qquad (62)$$

and

$$\langle a(\mathbf{k})a^\dagger(\mathbf{q})\rangle = \frac{\delta^{(3)}(\mathbf{k} - \mathbf{q})}{V}\langle a(\mathbf{k})a^\dagger(\mathbf{k})\rangle,$$

$$\langle a^\dagger(\mathbf{k})a(\mathbf{q})\rangle = \frac{\delta^{(3)}(\mathbf{k} - \mathbf{q})}{V}\langle a^\dagger(\mathbf{k})a(\mathbf{k})\rangle. \tag{63}$$

Using these expressions, the integrals in Eq. (61) are trivially performed and we obtain

$$i\Delta_0^>(p) = \frac{\pi}{E_p}\delta(E_p - p_0)\frac{1}{V}\langle a(\mathbf{p})a^\dagger(\mathbf{p})\rangle$$

$$+ \frac{\pi}{E_p}\delta(E_p + p_0)\frac{1}{V}\langle a^\dagger(-\mathbf{p})a(-\mathbf{p})\rangle. \tag{64}$$

Using the commutation relation $[a(\mathbf{p}), a^\dagger(\mathbf{k})] = \delta^{(3)}(\mathbf{p} - \mathbf{k})$, which gives

$$\langle a(\mathbf{k})a^\dagger(\mathbf{k})\rangle = \langle a^\dagger(\mathbf{k})a(\mathbf{k})\rangle + V,$$

one finds that Eq. (64) combined with the definition (59) reproduces (55). Thus we have shown that the definitions (51) and (59) are consistent with each other for homogeneous systems. One notes that for nonhomogeneous systems the conditions (62) and (63) are, in principle, not fulfilled. As a result, the functions $\Delta_0^{\lessgtr}(X, p)$ are not only X-dependent but also have support for off-shell momenta p in agreement with Eq. (34). Therefore, the expressions (54) and (55) can be treated as a representation of $\Delta_0^{\lessgtr}(X, p)$ only for weakly nonhomogeneous systems.

X. MASS-SHELL CONSTRAINT

In this section we discuss two issues that are related to the mass-shell constraint. We first show that in the homogeneous limit (or neglecting gradient terms), the mass-shell condition is satisfied if the distribution function is defined in terms of the spectral function as in Eq. (51), and if this distribution function satisfies the transport equation. Secondly, we show that using this definition of the distribution function, the mass-shell constraint reduces to the familiar definition of the mass-shell condition on the four-momentum in the limit of zero-width quasiparticles.

The transport and mass-shell equations for a homogeneous system, which are obtained from Eqs. (35) and (36) by dropping gradient terms, read

$$\Pi^>(X, p)\Delta^<(X, p) - \Pi^<(X, p)\Delta^>(X, p) = 0, \tag{65}$$

$$[p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p)]\Delta^{\lessgtr}(X, p)$$

$$= -\Pi^{\lessgtr}(X, p)\mathrm{Re}\Delta^+(X, p). \tag{66}$$

These equations can be rewritten using Eqs. (52) and (53) which express $\Delta^<(X, p)$ through the distribution function. We work below with the mass-shell equation for $\Delta^<(X, p)$ and restrict to $p_0 > 0$. It is straightforward to repeat all steps with $p_0 < 0$, and for $\Delta^<$ with $p_0 > 0$ and $p_0 < 0$. We find

$$A(x, p)[\Pi^>(X, p)f(X, p) - \Pi^<(X, p)(f(X, p) + 1)] = 0, \tag{67}$$

$$[p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p)]A(X, p)f(X, p)$$

$$= -i\Pi^<(X, p)\mathrm{Re}\Delta^+(X, p). \tag{68}$$

Using the spectral function Eq. (47), the mass-shell equation (68) has the form

$$\mathrm{Re}\Delta^+(x, p)[\Pi^>(X, p)f(X, p) - \Pi^<(X, p)(f(X, p) + 1)]$$

$$= 0. \tag{69}$$

One sees that if f solves Eq. (67), it automatically satisfies Eq. (69). Equivalently, the transport and mass-shell equations (35) and (36) can be replaced with the two Eqs. (46) and (67) in the homogeneous limit.

We note that the Eqs. (67) and (68), which were obtained by neglecting gradient terms, are exact for equilibrium (homogeneous) systems. These equations can be written as

$$\Pi^>(p)f^{\mathrm{eq}}(p) = \Pi^<(p)(f^{\mathrm{eq}}(p) + 1), \tag{70}$$

which is the well-known Kubo-Martin-Schwinger (KMS) condition, see, e.g., [13], that is satisfied by the self-energy in equilibrium. We also note that in going from Eqs. (35) and (36) to Eqs. (47) and (67), we have replaced four equations with two. This reduction in the number of independent Green's functions is expected since in equilibrium the two functions $\Delta^{\lessgtr}(X, p)$ are not independent but related through a KMS condition of the form (70).

We have shown above that when the distribution function is defined by Eq. (51), the transport and mass-shell equations (35) and (36) can be replaced by the Eqs. (47) and (67). In the limit $\mathrm{Im}\Pi^+ \to 0$ the solution to Eq. (47), which is given by Eq. (48), becomes

$$A(X, p) = 2\pi(\Theta(p_0) - \Theta(-p_0))$$

$$\times \delta(p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p)). \tag{71}$$

The argument of the delta function gives the usual form of the mass-shell constraint of the four-momentum

$$p^2 - m^2 + \Pi_\delta(X) + \mathrm{Re}\Pi^+(X, p) = 0, \tag{72}$$

which tell us that only three out of four-momentum components are independent. For $\mathrm{Im}\Pi^+$ small but finite, this equation determines the position of the maximum of the spectral function.

We note that when the system of interest is significantly inhomogeneous cf. Eq. (34), or when an interaction generates a non-negligible value of $\mathrm{Im}\Pi^+$, the function $\Delta^{\lessgtr}(X, p)$, which solves Eq. (36), has support from momenta not satisfying the relation (72). In this case, one has finite-width quasiparticles and the Eq. (72) only gives, according to Eq. (48), the most probable energy of a quasiparticle. Thus, the statement that the four-momentum is on the mass-shell (72) and the statement that the function $\Delta^{\lessgtr}(X, p)$ satisfies the mass-shell constraint (33) are, in general, not equivalent.

XI. TRANSPORT EQUATION

The distribution function f satisfies the transport equation which can be obtained from Eq. (35) for $\Delta^>$ or $\Delta^<$. Using Eqs. (46) and (51), one finds

$$\Theta(p_0)A(X, p)\{p^2 - m^2 + \Pi_\delta(X) + \text{Re}\Pi^+(X, p), f(X, p)\} = i\Theta(p_0)[A(X, p)\Big(\Pi^>(X, p)f(X, p) - \Pi^<(X, p)(f(X, p) + 1)\Big)$$
$$+ f(X, p)\{\Pi^>(X, p), \text{Re}\Delta^+(X, p)\}$$
$$- (f(X, p) + 1)\{\Pi^<(X, p), \text{Re}\Delta^+(X, p)\}], \qquad (73)$$

where we have used the identity

$$\{A, BC\} = \{A, B\}C + \{A, C\}B.$$

Neglecting the gradient terms in the right-hand side of Eq. (73), we obtain the equation

$$\Theta(p_0)\{p^2 - m^2 + \Pi_\delta(X) + \text{Re}\Pi^+(X, p), f(X, p)\}$$
$$= i\Theta(p_0)[\Pi^>(X, p)f(X, p) - \Pi^<(X, p)(f(X, p) + 1)]. \qquad (74)$$

The transport Eq. (74) greatly simplifies for zero-width quasiparticles. Using the mass-shell condition (72), and taking the positive energy solution, one can evaluate the Poisson-like bracket to obtain

$$\Theta(p_0)\{p^2 - m^2 + \Pi_\delta(X) + \text{Re}\Pi^+(X, p), f(X, \mathbf{p})\}$$
$$= 2E_p\Big(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\Big)f(X, \mathbf{p}) + 2\nabla V(X, p)\nabla_p f(X, \mathbf{p}), \qquad (75)$$

where

$$V(X, p) \equiv \Pi_\delta(X) + \text{Re}\Pi^+(X, p),$$

and the velocity $\mathbf{v} \equiv \partial E_p/\partial \mathbf{p}$. Note that E_p is not $\sqrt{\mathbf{p}^2 + m^2}$ but the positive solution of Eq. (72). Substituting Eq. (75) into Eq. (74), the transport equation takes the form

$$E_p\Big(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\Big)f(X, \mathbf{p}) + \nabla V(X, \mathbf{p})\nabla_p f(X, \mathbf{p})$$
$$= \Theta(p_0)\frac{i}{2}[\Pi^>(X, p)f(X, \mathbf{p})$$
$$- \Pi^<(X, p)(f(X, \mathbf{p}) + 1)]. \qquad (76)$$

In the left-hand side of Eq. (76), one recognizes the standard drift and Vlasov terms [37]. The calculation of the self-energies $\text{Re}\Pi^+$ and Π_δ in the Vlasov term is straightforward and the physical interpretation is well known: they contribute to mean-field effects and result in the generation of an effective mass. In this paper we will not discuss these terms (they have been studied in detail in [18]). From the right-hand side of Eq. (76), one defines the collision term $C(X, p)$ as

$$\Theta(p_0)C(X, p) \overset{\text{def}}{=} \Theta(p_0)\frac{i}{2}[\Pi^>(X, p)f(X, \mathbf{p})$$
$$- \Pi^<(X, p)(f(X, \mathbf{p}) + 1)]. \qquad (77)$$

The structure of this collision term is the subject of the next five sections.

XII. COLLISION TERM STRUCTURE

In order to obtain physical results from the transport equation derived in the previous sections, we must specify the self-energies that appear in the collision term. We will assume that the interaction is sufficiently weak that the self-energies can be expanded in powers of the coupling constant, and thus expressed through the Green's functions. We will develop a diagrammatic expansion of the self-energies and identify the different physical processes that contribute to the collision term (77). The collision terms we obtain are local in space-time, and consequently valid in the Markovian limit where all memory effects are neglected.

We first rewrite Eq. (77) by multiplying by the spectral function $A(X, p)$ and using Eq. (51). Thus, we obtain the equation

$$\Theta(p_0)A(X, p)C(X, p) = \Theta(p_0)\tfrac{1}{2}(\Pi^<(X, p)\Delta^>(X, p)$$
$$- \Pi^>(X, p)\Delta^<(X, p)). \qquad (78)$$

Evolution along the contour illustrated in Fig. 1 is formally very similar to evolution along the real time axis, and consequently it is possible to define perturbation theory on the contour. Self-energies with real time arguments, in particular $\Pi^<_>$, have to be extracted from the contour self-energy. Since these calculations involve summations over contributions from both branches of the contour, they are much more difficult than their vacuum counterparts. There are several different methods that can be used which are based on different representations of real time field theory [38] and each has its advantages and proponents. At the lower loop levels, the relevant self-energies have been calculated in several different ways both in equilibrium [39] and nonequilibrium [18]. (A general structure of the equilibrium self-energy expressed through the scattering amplitude has been studied in

[40].) At three or more loops, the calculation is extremely cumbersome and requires the use of special methods.

Our calculation of the self-energies, which enter the collision term of transport equation, uses the Keldysh representation formulated in terms of retarded, advanced and symmetric Green functions. To understand why the Keldysh representation is better suited to perform such a calculation, we need to look at the Wigner transformed propagators for noninteracting fields in the Keldysh and $1/2$ representation. In the $1/2$ basis the propagators are given by Eqs. (54)–(57) and in the Keldysh basis by (43) and (58). The functions $i\Delta_0^<$ and $i\Delta_0^>$ do not contain off-mass-shell contributions that have the form of propagators. They are nonzero only on the mass shell. In contrast, $i\Delta_0^c$ and $i\Delta_0^a$ contain one piece that corresponds to time-ordered and anti-time-ordered propagation, respectively, and one piece that is nonzero only on the mass shell. In the Keldysh basis, however, none of the Green's functions mixes the propagating pieces and terms that are nonzero only on the mass shell: Δ_0^+ and Δ_0^- contain only the contributions corresponding to retarded and advanced propagation, respectively, and Δ_0^{sym} is nonzero only on-mass-shell. Remembering that on-mass-shell Green functions represent real particles and propagating Green functions virtual particles, it is not unexpected that this clean separation of real and virtual particles makes the physical interpretation of the collision term much more straightforward.

We switch to the Keldysh basis using the relations (14) and analogous relations for the self-energies. The collision term (78), which will be computed in the next sections up to four-loop level, then equals

$$\Theta(p_0)A(p)C(p) = \tfrac{1}{4}\Theta(p_0)[\Pi^{\text{sym}}(p)(\Delta^+(p) - \Delta^-(p))$$
$$- (\Pi^+(p) - \Pi^-(p))\Delta^{\text{sym}}(p)]. \quad (79)$$

It will be shown that in the Keldysh representation, all contributions to the collision term have the form of "cuts" [27]. We use the word "cut" in the way that it is usually used when discussing the calculation of the imaginary part of a diagram: a "cut line" passes through a diagram from top to bottom dividing it into two parts, so that each part contains at least one external leg. A propagator that is crossed by the cut line becomes a "cut propagator" which means that it has been put on the mass shell.

We introduce the notion "central cut" to refer to the cut for which all loops contain at least one cut propagator. For the one-loop diagram in ϕ^3 theory, and the two-loop diagram in ϕ^4 theory, the only possible cut is the central cut. At two and more loops in ϕ^3 theory, and three and more loops in ϕ^4 theory, all diagrams contain contributions from noncentral cuts. In addition, there are some diagrams for which no central cut exists. In each case however, the contributions from noncentral cuts can be included as renormalized lower loop terms. An example of this is presented in Sec. XIV B. The main thrust of our paper

involves the analysis of the central cuts of three- and four-loop diagrams.

The calculation of a self-energy graph involves two nontrivial steps: the calculation of the integrand and the computation of the integral itself. In this paper we are primarily interested in the calculation of integrands which determine the structure of the collision term. At three or more loops, calculating by hand in any representation is extremely tedious. We use a MATHEMATICA program developed by one of us, which is described in detail in [26], to calculate the integrand. To use the program one assigns momenta and indices to every line and vertex in the diagram. Each index can take the values 1 or 2, corresponding to the top and bottom branches of the contour. The indices that correspond to external legs take fixed values, and a specific combination corresponds to a specific self-energy in the $1/2$ representation, in analogy with Eq. (8). Internal indices are summed over their two possible values 1 and 2. Using Eqs. (8), (13), and (14) one switches to the Keldysh representation. A huge number of terms is produced even for fairly simple diagrams. However, there are many cancellations between these terms. The program identifies the surviving contributions and provides them as output.

In Secs. XIII and XIV we rederive the well-known one- and two-loop results to illustrate our technique. In Sec. XV we use our method to obtain results for three-loop diagrams which contribute to $2 \leftrightarrow 3$ processes in ϕ^3 theory, and in Sec. XVI we study four-loop diagrams in ϕ^4 theory which correspond to $2 \leftrightarrow 4$ and $3 \leftrightarrow 3$ processes.

XIII. ONE-LOOP CONTRIBUTIONS

We start by considering the lowest order contributions. In the ϕ^4 model, the only one-loop diagram is the tadpole. It is well known that this diagram only contributes to mean-field dynamics, see, e.g., [18], and therefore it will not be discussed here. For ϕ^3 theory, the one-loop contribution to the self-energy, which is shown in Fig. 2, has a nontrivial imaginary part. The related scattering amplitude corresponds to $1 \leftrightarrow 2$ processes which are kinematically forbidden for on-mass-shell particles, as an on-shell particle of mass m cannot decay into two on-shell particles with the same masses. However, it is important to remember that the process is allowed for virtual particles and has been repeatedly discussed, in particular, in the context of
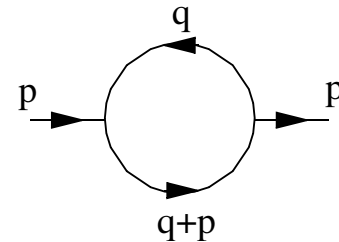

FIG. 2.   One-loop contribution to the ϕ^3 self-energy.

electron-positron-photon interactions. We will discuss this contribution in detail to illustrate the method that we will use in the analysis of more complicated multiloop diagrams.

At this point, we introduce some notational simplifications. We suppress the X-dependence of all functions. All Green's functions used in our perturbative calculations correspond to free fields, and thus we suppress the index "0" on these functions. Similarly, we suppress from now on the index 0 which has been used to denote the free distribution function and the free spectral function. Finally, we define on-shell four-momenta

$$\tilde{p} \equiv (E_p, \mathbf{p}), \qquad (80)$$

with $E_p \equiv \sqrt{\mathbf{p}^2 + m^2}$. We note that $\sqrt{\mathbf{p}^2 + m^2}$ should be denoted E_p^0, but as above the index 0 is suppressed.

The Keldysh self-energies corresponding to the diagram from Fig. 2 are

$$\Pi^+(p) = i\frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} [\Delta^-(q)\Delta^{\text{sym}}(q + p) + \Delta^{\text{sym}}(q)\Delta^+(q + p)],$$

$$\Pi^-(p) = i\frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} [\Delta^{\text{sym}}(q)\Delta^-(q + p) + \Delta^+(q)\Delta^{\text{sym}}(q + p)], \qquad (81)$$

$$\Pi^{\text{sym}}(p) = i\frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} [\Delta^{\text{sym}}(q)\Delta^{\text{sym}}(q + p) + \Delta^+(q)\Delta^-(q + p) + \Delta^-(p)\Delta^+(q + p)].$$

We substitute these self-energies into the right-hand side of the collision term (79) and express $\Delta^{\text{sym}}(p)$ through the distribution function according to Eq. (58). We also use Eqs. (45) and (50) which give

$$i(\Delta^+(p) - \Delta^-(p)) = \text{sgn}(p_0)2\pi\delta(p^2 - m^2), \qquad (82)$$

and we observe that

$$\int \frac{d^4q}{(2\pi)^4} \Delta^+(q)\Delta^+(q + p) = \int \frac{d^4q}{(2\pi)^4} \Delta^-(q)\Delta^-(q + p) = 0, \qquad (83)$$

due to the positions of poles of $\Delta^\pm$. Thus, we obtain

$$\Theta(p_0)A(p)C_{1L-\phi^3}(p) = \Theta(p_0)\frac{g^2}{16} \int \frac{d^4q}{(2\pi)^4} \sum_{\{n_q, n_p, n_r\}=\pm 1} \mathcal{F}_3(n_p, \mathbf{p}; n_q, \mathbf{q}; -n_r, -\mathbf{r})$$

$$\times \prod_{\alpha=\{p,q,r\}} n_\alpha \text{sgn}(\alpha_0)2\pi\delta(\alpha^2 - m^2)\Theta(n_\alpha\alpha_0), \qquad (84)$$

where $r \equiv q + p$ and α is a generic four-momentum variable which equals p, q or r, and $n_\alpha = \pm 1$. The statistical factor $\mathcal{F}_N$ is defined as

$$\mathcal{F}_N(n_1, \mathbf{p}_1; n_2, \mathbf{p}_2; \cdots n_N, \mathbf{p}_N)$$

$$\equiv \Omega(n_1, \mathbf{p}_1)\Omega(n_2, \mathbf{p}_2)\cdots\Omega(n_N, \mathbf{p}_N)$$

$$- \Omega(-n_1, -\mathbf{p}_1)\Omega(-n_2, -\mathbf{p}_2)\cdots\Omega(-n_N, -\mathbf{p}_N) \qquad (85)$$

with

$$\Omega(n, \mathbf{p}) \equiv (1 + n)(1 + f(\mathbf{p})) + (1 - n)f(-\mathbf{p}). \qquad (86)$$

We note that the asymmetry in the signs of $(n_q, \mathbf{q})$ and $(n_r, \mathbf{r})$ in the factor $\mathcal{F}_3(n_p, \mathbf{p}; n_q, \mathbf{q}; -n_r, -\mathbf{r})$ occurs because the momenta q and r are defined to flow in opposite directions in Fig. 2.

To further simplify our result, we write the delta functions as

$$\text{sgn}(\alpha_0)\delta(\alpha^2 - m^2) = \frac{1}{2E_\alpha} \sum_{n_\alpha=\pm 1} n_\alpha\Theta(n_\alpha\alpha_0)$$

$$\times \delta(\alpha_0 - n_\alpha E_\alpha), \qquad (87)$$

where $E_\alpha \equiv \sqrt{m^2 + \alpha^2}$, and get

$$\Theta(p_0)A(p)C_{1L-\phi^3}(p) = \Theta(p_0)\frac{g^2\pi^3}{16} \int \frac{d^4q}{(2\pi)^4} \sum_{\{n_p, n_q, n_r\}=\pm 1} \mathcal{F}_3(n_p, \mathbf{p}; n_q, \mathbf{q}; -n_r, -\mathbf{r}) \prod_{\alpha=\{p,q,r\}} \frac{1}{E_\alpha}\delta(\alpha_0 - n_\alpha E_\alpha). \qquad (88)$$

We rewrite the expression (88) by introducing the delta function $\delta^{(4)}(p + q - r)$ and an integral over r, which allows us to treat r as an independent variable. Performing the integrals over p_0 and r_0, one obtains

$$\Theta(p_0)A(p)C_{1L-\phi^3}(p) = \Theta(p_0)\frac{g^2\pi}{64}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{r})\prod_{\alpha=\{p,q,r\}}\frac{1}{E_\alpha}\sum_{\{n_p,n_q,n_r\}=\pm1}\delta(p_0-n_pE_p)$$

$$\times 2\pi\delta(n_pE_p+n_qE_q-n_rE_r)\mathcal{F}_3(n_p,\mathbf{p};n_q,\mathbf{q};-n_r,-\mathbf{r}) \quad (89)$$

We discuss below the physical interpretation of the positive energy piece of the collision term (89) which will be denoted with the index (+). We drop the theta function in Eq. (89) and take only the $n_p = 1$ term in the sum. We also use the explicit form of the free spectral function (50) to make the replacement

$$\Theta(p_0)A(p) = \frac{\pi}{E_p}\delta(p_0-E_p). \quad (90)$$

In principle, we are left with four terms that come from the sum over n_q and n_r in Eq. (89). One of these terms corresponds to a delta function of the form $\delta(E_p + E_q + E_r)$ which has no support. The remaining three terms correspond to the processes symbolically denoted as: $p \leftrightarrow q + r$, $p + q \leftrightarrow r$ and $p + r \leftrightarrow q$. The last two terms can be combined by making the variable transformation $q \leftrightarrow r$ in the second term.

We consider first the contribution to the collision term (89) obtained from choosing $n_q = -1$ and $n_r = 1$. Making the change of variables $\mathbf{q} \to -\mathbf{q}$, we find

$$C_{1L-\phi^3}^{(+)}[p \leftrightarrow q + r] = \frac{g^2}{64}\int\frac{d^3q}{(2\pi)^3E_q}\int\frac{d^3r}{(2\pi)^3E_r}$$

$$\times (2\pi)^4\delta^{(4)}(\tilde{p}-\tilde{q}-\tilde{r})$$

$$\times \mathcal{F}_3(1,\mathbf{p};-1,-\mathbf{q};-1,-\mathbf{r}), \quad (91)$$

where the statistical factor equals

$$\mathcal{F}_3(1,\mathbf{p};-1,-\mathbf{q};-1,-\mathbf{r}) = 2^3[(1+f(\mathbf{p}))f(\mathbf{q})f(\mathbf{r})$$

$$- f(\mathbf{p})(1+f(\mathbf{q}))$$

$$\times (1+f(\mathbf{r}))]. \quad (92)$$

This process is represented in Fig. 3.

Next, we look at the contributions to Eq. (89) obtained from choosing $n_q = n_r = 1$ and $n_q = n_r = -1$. After changing variables and combining the two terms, we get

$$C_{1L-\phi^3}^{(+)}[p + q \leftrightarrow r] = \frac{g^2}{32}\int\frac{d^3q}{(2\pi)^3E_q}\int\frac{d^3r}{(2\pi)^3E_r}$$

$$\times (2\pi)^4\delta^{(4)}(\tilde{p}+\tilde{q}-\tilde{r})$$

$$\times \mathcal{F}_3(1,q;1,\mathbf{q};-1,-\mathbf{r}), \quad (93)$$

with

$$\mathcal{F}_3(1,\mathbf{p};1,\mathbf{q};-1,-\mathbf{r}) = 2^3[(1+f(\mathbf{p}))(1+f(\mathbf{q}))f(\mathbf{r})$$

$$- f(\mathbf{p})f(\mathbf{q})(1+f(\mathbf{r}))]. \quad (94)$$

This contribution is illustrated in Fig. 4.

The complete one-loop collision term equals

$$C_{1L-\phi^3}^{(+)}(p) = \int\frac{d^3q}{(2\pi)^3E_q}\int\frac{d^3r}{(2\pi)^3E_r}|\mathcal{M}|_{1L-\phi^3}^2\left[(2\pi)^4\delta^{(4)}(\tilde{p}-\tilde{q}-\tilde{r})\frac{1}{2!}[(1+f(\mathbf{p}))f(\mathbf{q})f(\mathbf{r})-f(\mathbf{p})(1+f(\mathbf{q}))(1+f(\mathbf{r}))]\right.$$

$$\left. + (2\pi)^4\delta^{(4)}(\tilde{p}+\tilde{q}-\tilde{r})[(1+f(\mathbf{p}))(1+f(\mathbf{q}))f(\mathbf{r})-f(\mathbf{p})f(\mathbf{q})(1+f(\mathbf{r}))]\right], \quad (95)$$

where the scattering matrix element equals $|\mathcal{M}|_{1L-\phi^3}^2 = g^2/4$. The two terms proportional to $f(\mathbf{p})$ correspond to "loss" contributions and the terms proportional to $(1 + f(\mathbf{p}))$ correspond to "gain" contributions. In the first term, which represents the process $p \leftrightarrow q + r$, there is the factor $1/2!$ because of the integration over momenta of two identical particles in either final or initial scattering state.

We remind the reader that the contribution (95) to the collision term is identically zero because it is impossible to satisfy the delta function constraint. Physically, an on-shell particle of mass m cannot decay into two on-shell particles
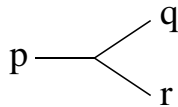
with the same masses. We have presented this calculation to illustrate the method that we will use in the analysis of more complicated multiloop diagrams.

XIV. TWO-LOOP CONTRIBUTIONS

First we note that at any number of loops greater than one, there are two different kinds of diagrams that contribute to the self-energy. The first class contains diagrams that are built from diagrams of lower loop order by adding tadpole type insertions. An example is shown in Fig. 5.

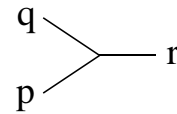FIG. 3. $p \leftrightarrow q + r$ scattering process.

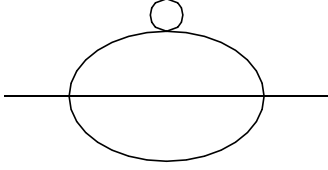FIG. 4. $p + q \leftrightarrow r$ scattering process.

FIG. 5.   Diagram with a tadpole insertion.

The contribution to the collision term from a diagram of this type has the same structure as the corresponding lower loop diagram with the addition of effective masses coming from the tadpole insertions. This type of diagram will not be discussed here. The second class of diagrams does not contain tadpoles. This is the type of diagram that we study in this paper. For the first few diagrams we give some of the details of the calculation but for the most part we focus on the final results.

At the two-loop level, there are two contributions to the self-energy in ϕ^3 theory and one contribution in ϕ^4 theory (excluding diagrams with tadpoles).

A. ϕ^4 theory

The two-loop contribution to self-energy is shown in Fig. 6. As in the case of the one-loop contribution in ϕ^3 theory, there is only one cut which is the central one. Calculating the retarded, advanced, and symmetric components of the self-energy, we obtain

$$\Pi^+(p) = \frac{g^2}{24} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} [\Delta^-(q)(\Delta^{\text{sym}}(l)\Delta^{\text{sym}}(s)$$
$$+ \Delta^-(l)\Delta^+(s)) + \Delta^{\text{sym}}(q)(\Delta^-(l)\Delta^{\text{sym}}(s)$$
$$+ \Delta^{\text{sym}}(l)\Delta^+(s))], \qquad (96)$$
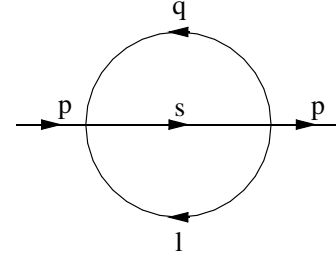
FIG. 6.   Two-loop contribution to the ϕ^4 self-energy.

$$\Pi^-(p) = \frac{g^2}{24} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} [\Delta^{\text{sym}}(s)(\Delta^{\text{sym}}(q)\Delta^+(l)$$
$$+ \Delta^{\text{sym}}(l)\Delta^+(q)) + \Delta^-(s)(\Delta^{\text{sym}}(l)\Delta^{\text{sym}}(q)$$
$$+ \Delta^+(l)\Delta^+(q))], \qquad (97)$$

$$\Pi^{\text{sym}}(p) = \frac{g^2}{24} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} [\Delta^+(l)(\Delta^-(s)\Delta^{\text{sym}}(q)$$
$$+ \Delta^{\text{sym}}(s)\Delta^+(q)) + \Delta^-(l)(\Delta^-(q)\Delta^{\text{sym}}(s)$$
$$+ \Delta^{\text{sym}}(q)\Delta^+(s)) + \Delta^{\text{sym}}(l)(\Delta^{\text{sym}}(q)\Delta^{\text{sym}}(s)$$
$$+ \Delta^-(s)\Delta^+(q) + \Delta^-(q)\Delta^+(s))], \qquad (98)$$

where we have used $s \equiv p + q + l$. We substitute these formulas into the collision term (79) and express the symmetric Green's function through the distribution function according to Eq. (58). As in the case of the one-loop diagram, we use Eqs. (82) and (83) to find

$$\Theta(p_0)A(p)C_{2L-\phi^4}(p_0) = \Theta(p_0)\frac{g^2\pi^4}{192} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \sum_{\{n_p,n_q,n_l,n_s\}=\pm 1} \mathcal{F}_4(n_p, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; -n_s, -\mathbf{s})$$

$$\times \prod_{\alpha=\{p,q,ls\}} \frac{1}{E_\alpha} \delta(\alpha_0 - n_\alpha E_\alpha). \qquad (99)$$

We rewrite the result (99) in a more symmetric way by introducing the delta function $\delta^{(4)}(p + q + l - s)$ and an integral over s, which allows us to treat s as an independent variable. As in the previous section, we use the free spectral function (90) and consider only positive energy contributions to the collision term. Performing the integrals over q_0, l_0, and s_0, we have

$$C^{(+)}_{2L-\phi^4}(p) = \frac{g^2\pi^3}{192} \frac{1}{(2\pi)^8} \int d^3q \int d^3l \int d^3s \, \delta^{(3)}(\mathbf{p} + \mathbf{q} - \mathbf{l} - \mathbf{s}) \frac{1}{E_q E_l E_s}$$

$$\times \sum_{\{n_p,n_l,n_s\}=\pm 1} \delta(E_p + n_q E_q + n_l E_l - n_s E_s)\mathcal{F}_4(1, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; -n_s, -\mathbf{s}). \qquad (100)$$

In principle the sum produces eight terms, but five of these eight terms are zero because the corresponding energy conserving delta function has no support. There are three choices of $\{n_q, n_l, n_s\}$ that give nonzero contributions to the collision integral. These three terms can be combined by making an appropriate variable transformation on each

term. We list below: the three choices for the n_i variables with $i = \{q, l, s\}$ that give nonzero contributions, the variable transformations for each of these three terms that allow us to combine them, and the corresponding transformations to on-shell four-momenta [using the notation (80)]:

$$[1]\ n_q = -n_l = n_s = 1; \qquad \mathbf{l} \to -\mathbf{l}; \qquad q \to \tilde{q},\ l \to -\tilde{l},\ s \to \tilde{s};$$

$$[2]\ n_q = -n_1 = -n_s = -1; \qquad \mathbf{q} \to -\mathbf{l},\ \mathbf{l} \to \mathbf{q}; \qquad q \to -\tilde{l},\ l \to \tilde{q},\ s \to \tilde{s}; \qquad (101)$$

$$[3]\ n_q = n_1 = n_8 = -1; \qquad \mathbf{q} \to -\mathbf{s},\ \mathbf{l} \to -\mathbf{l},\ s \to -\mathbf{q}; \qquad q \to -\tilde{s},\ l \to -\tilde{l},\ s \to -\tilde{q}.$$

The statistical factor and the two delta functions have the same form for each term. Combining the deltas into one four-dimensional delta function that expresses energy-momentum conservation and using the explicit form of the statistical factor defined by Eqs. (85) and (86), we finally find

$$C^{(+)}_{2L-\phi^4}[p + q \leftrightarrow l + s] = \frac{1}{2!} \int \frac{d^3 q}{(2\pi)^3 E_q} \int \frac{d^3 l}{(2\pi)^3 E_l} \int \frac{d^3 s}{(2\pi)^3 E_s} (2\pi)^4 \delta^{(4)}(\tilde{q} + \tilde{p} - \tilde{l} - \tilde{s})$$

$$\times |\mathcal{M}|^2_{2L-\phi^4}[(1 + f(\mathbf{p}))(1 + f(\mathbf{q}))f(\mathbf{l})f(\mathbf{s}) - f(\mathbf{p})f(\mathbf{q})(1 + f(\mathbf{l}))(1 + f(\mathbf{s}))], \qquad (102)$$

where $|\mathcal{M}|^2_{2L-\phi^4} = g^2/16$. Equation (102) gives the standard collision term corresponding to the binary process symbolically denoted as $p + q \leftrightarrow l + s$ and illustrated in Fig. 7. The first (second) contribution is the usual gain (loss) term.

B. ϕ^3 theory

In ϕ^3 theory, there are two graphs of different topologies that contribute at two-loop order as shown in Fig. 8.

We begin with a detailed calculation of the contribution to the collision term from the first diagram in Fig. 11. The results for the self-energies are as follows

$$\Pi^+(p) = \frac{g^4}{8} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} [\Delta^-(q)\Delta^-(r)\Delta^{\mathrm{sym}}(r)(\Delta^-(l)\Delta^{\mathrm{sym}}(k) + \Delta^{\mathrm{sym}}(l)\Delta^+(k))$$

$$+ \Delta^-(q)\Delta^-(r)\Delta^+(r)(\Delta^{\mathrm{sym}}(k)\Delta^{\mathrm{sym}}(l) + \Delta^-(l)\Delta^+(k) + \Delta^-(k)\Delta^+(l))$$

$$+ \Delta^-(q)\Delta^{\mathrm{sym}}(r)\Delta^+(r)(\Delta^-(k)\Delta^{\mathrm{sym}}(l) + \Delta^{\mathrm{sym}}(k)\Delta^+(l)) + \Delta^{\mathrm{sym}}(q)\Delta^+(r)^2(\Delta^-(k)\Delta^{\mathrm{sym}}(l)$$

$$+ \Delta^{\mathrm{sym}}(k)\Delta^+(l))], \qquad (103)$$

$$\Pi^-(p) = \frac{g^4}{8} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} [(\Delta^-(r))^2 \Delta^{\mathrm{sym}}(q)(\Delta^-(l)\Delta^{\mathrm{sym}}(k) + \Delta^{\mathrm{sym}}(l)\Delta^+(k))$$

$$+ \Delta^+(r)\Delta^-(r)\Delta^{\mathrm{sym}}(r)\Delta^-(l)\Delta^{\mathrm{sym}}(r)(\Delta^-(l)\Delta^{\mathrm{sym}}(k) + \Delta^{\mathrm{sym}}(l)\Delta^+(k)) + \Delta^+(r)\Delta^-(r)\Delta^{\mathrm{sym}}(r)(\Delta^-(l)\Delta^{\mathrm{sym}}(k)$$

$$+ \Delta^{\mathrm{sym}}(l)\Delta^+(k)) + \Delta^-(r)\Delta^+(p)\Delta^+(r)(\Delta^{\mathrm{sym}}(k)\Delta^{\mathrm{sym}}(k)\Delta^{\mathrm{sym}}(l) + \Delta^-(l)\Delta^+(k) + \Delta^-(k)\Delta^+(l))$$

$$+ \Delta^{\mathrm{sym}}(r)\Delta^{\mathrm{sym}}(r)\Delta^+(p)\Delta^+(r)(\Delta^-(k)\Delta^{\mathrm{sym}}(l) + \Delta^-(l)\Delta^+(k) + \Delta^-(k)\Delta^+(l))$$

$$+ \Delta^{\mathrm{sym}}(r)\Delta^+(p)\Delta^+(r)(\Delta^-(k)\Delta^{\mathrm{sym}}(l) + \Delta^{\mathrm{sym}}(k)\Delta^+(l))], \qquad (104)$$

$$\Pi^{\mathrm{sym}}(p) = \frac{g^4}{8} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} [\Delta^-(r)\Delta^{\mathrm{sym}}(r)\Delta^{\mathrm{sym}}(q)(\Delta^-(l)\Delta^{\mathrm{sym}}(k) + \Delta^{\mathrm{sym}}(l)\Delta^+(k))$$

$$+ \Delta^+(r)\Delta^{\mathrm{sym}}(r)\Delta^{\mathrm{sym}}(q)(\Delta^-(k)\Delta^{\mathrm{sym}}(l) + \Delta^{\mathrm{sym}}(l) + \Delta^{\mathrm{sym}}(k)\Delta^+(l)) + (\Delta^-(r))^2\Delta^+(q)(\Delta^-(l)\Delta^{\mathrm{sym}}(k)$$

$$+ \Delta^{\mathrm{sym}}(l)\Delta^+(k)) + (\Delta^+(r))^2\Delta^-(q)(\Delta^-(k)\Delta^{\mathrm{sym}}(l) + \Delta^{\mathrm{sym}}(k)\Delta^+(l)) + \Delta^-(r)\Delta^+(r)\Delta^{\mathrm{sym}}(q)(\Delta^{\mathrm{sym}}(k)\Delta^{\mathrm{sym}}(l)$$

$$+ \Delta^-(l)\Delta^+(k) + \Delta^-(k)\Delta^+(l))]. \qquad (105)$$

We substitute these expressions into the collision term (79) and divide the results into contributions of three different types, which correspond to contributions to the three cuts shown in Fig. 9.
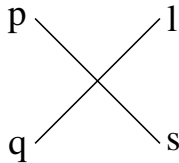

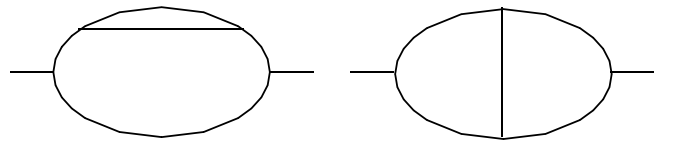FIG. 7. $p + q \leftrightarrow l + s$ scattering process.

FIG. 8.   Contributions to the self-energy in ϕ^3 theory.

These cuts are referred to as the "central," "left" and "right" cuts, respectively. Terms that contain a factor $\Delta^+(r)\Delta^-(r)$ correspond to the central cut. The terms with $\Delta^+(r)$ but without $\Delta^-(r)$ are associated with the right cut, and terms that contain $\Delta^-(r)$ but not $\Delta^+(r)$ correspond to the left cut. The results read:

$$\Theta(p_0)A(p)C(p)|_{\text{central}} = \Theta(p_0)\frac{g^4\pi^4}{64}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}\Delta^+(r)\Delta^-(r)\sum_{\{n_p,n_q,n_l,n_s\}=\pm 1}\mathcal{F}_4(n_p,\mathbf{p};n_q,\mathbf{q};n_l,\mathbf{l};-n_s,-\mathbf{s})$$

$$\times\prod_{\alpha=\{p,q,l,s\}}\frac{1}{E_\alpha}\delta(\alpha_0-n_\alpha E_\alpha), \tag{106}$$

$$\Theta(p_0)A(p)C(p)|_{\text{right}} = \Theta(p_0)\frac{g^2\pi^3}{32}\int\frac{d^4q}{(2\pi)^4}\Delta^+(r)\Pi^+(r)\sum_{\{n_p,n_q,n_r\}=\pm 1}\mathcal{F}_3(n_p,\mathbf{p};n_q,\mathbf{q};-n_r,-\mathbf{r})$$

$$\times\prod_{\alpha=\{p,q,r\}}\frac{1}{E_\alpha}\delta(\alpha_0-n_\alpha E_\alpha), \tag{107}$$

$$\Theta(p_0)A(p)C(p)|_{\text{left}} = \Theta(p_0)\frac{g^2\pi^3}{32}\int\frac{d^4q}{(2\pi)^4}\Delta^-(r)\Pi^-(r)\sum_{\{n_p,n_q,n_r\}=\pm 1}\mathcal{F}_3(n_p,\mathbf{p};n_q,\mathbf{q};-n_r,-\mathbf{r})\prod_{\alpha=\{p,q,r\}}\frac{1}{E_\alpha}\delta(\alpha_0-n_\alpha E_\alpha).$$
$$\tag{108}$$

In Eqs. (107) and (108) the factors $\Pi^+(r)$ and $\Pi^-(r)$ refer to one-loop self-energies of the form shown in Fig. 2. Thus, we find that the noncentral cuts give contributions that can be understood as corrections to the one-loop result as shown in Fig. 10.

Noncentral cuts can always be treated in this manner, i.e., every noncentral cut provides a higher order correction to the central cut of some other diagram. As explained in the Introduction, we are only interested in the dominant contributions to any given physical process, and consequently we consider only central cuts from now on. We calculate contributions to the collision term at leading order in g for each physical process from the relevant sum of centrally cut self-energy diagrams at the same order of g. The collision term contains some phase space integrals, the statistical factor $\mathcal{F}_{\mathcal{N}}$ [Eq. (85)], and the square of a matrix element which is constructed from uncut propagators $\Delta^\pm$. At higher orders in g, there are many self-energy diagrams that have to be included, some of which have more than one central cut. It is a nontrivial technical problem to combine all of these terms into a result which can be written as the square of a matrix element. This structure is obtained by an appropriate choice of momentum labels for the internal lines of the self-energy diagrams. We proceed as follows:

(1) the same set of momentum variables is always assigned to the cut propagators (equivalently, each central cut has the same $\mathcal{F}$ factor);
(2) all possible permutations of momentum variables for the uncut lines are considered;
(3) for each diagram, all permutations are summed and the sum is normalized with the appropriate weight.

Using this strategy, we calculate the nine diagrams shown in Figs. 11 where we have made the following definitions for momentum variables:

$$r \equiv p + q, \qquad t \equiv q + l,$$
$$h \equiv p + l, \qquad s \equiv r + l. \tag{109}$$

We note that the second topology in Fig. 8 appears twice as often as the first topology (6 times as compared with three) because of the fact that we take only one of two possible diagonal cuts for these diagrams.
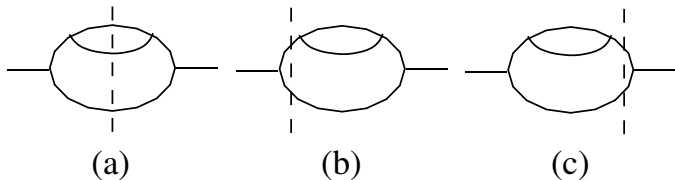
Defining the operation

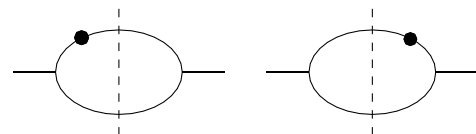FIG. 9. Cuts of the first diagram in Fig. 8.

FIG. 10. Noncentral cut for the first diagram in Fig. 8.

$$\ll \cdots \gg_{2L-\phi^3} \overset{\text{def}}{=} \Theta(p_0) \frac{g^4 \pi^4}{64} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \sum_{\{n_p, n_q, n_l, n_s\}=\pm 1} \mathcal{F}_4(n_p, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; -n_s, -\mathbf{s})$$

$$\times \prod_{\alpha=\{p,q,l,s\}} \frac{1}{E_\alpha} \delta(\alpha_0 - n_\alpha E_\alpha) \cdots, \tag{110}$$

the contributions corresponding to the diagrams from Figs. 11(a)–11(c) and 12(a)–12(f) can be written as

$$(11a) = \ll \Delta^+(r)\Delta^-(r) \gg_{2L-\phi^3}, \qquad (11b) = \ll \Delta^+(h)\Delta^-(h) \gg_{2L-\phi^3}, \qquad (11c) = \ll \Delta^+(t)\Delta^-(t) \gg_{2L-\phi^3},$$

$$(12a) = \ll \Delta^+(r)\Delta^+(t) \gg_{2L-\phi^3}, \qquad (12b) = \ll \Delta^+(r)\Delta^-(h) \gg_{2L-\phi^3}, \qquad (12c) = \ll \Delta^+(h)\Delta^+(t) \gg_{2L-\phi^3},$$

$$(12d) = \ll \Delta^+(h)\Delta^-(r) \gg_{2L-\phi^3}, \qquad (12e) = \ll \Delta^-(t)\Delta^-(r) \gg_{2L-\phi^3}, \qquad (12f) = \ll \Delta^-(t)\Delta^-(h) \gg_{2L-\phi^3}.$$

After summing these contributions we obtain,

$$\Theta(p_0)A(p)C_{2L-\phi^3}(p) = \tfrac{1}{3} \ll |\Delta^+(r) + \Delta^+(h) + \Delta^-(t)|^2 \gg_{2L-\phi^3}. \tag{111}$$

In order to express the result (111) in a more symmetric way, we introduce the delta function $\delta^{(4)}(p + q + l - s)$ and an integration over s, which allows us to treat s as an independent variable. As we have done previously, we use Eq. (90) and consider only positive energy contributions to the collision term. Performing the integrals over p_0, l_0 and s_0, we obtain

$$C_{2L-\phi^3}^{(+)}(p) = \frac{g^4 \pi^3}{192} \frac{1}{(2\pi)^8} \int d^3 q \int d^3 l \int d^3 s \, \delta^{(3)}(\mathbf{q} + \mathbf{p} + \mathbf{l} - \mathbf{s}) \frac{1}{E_q E_l E_s} |\Delta^+(p+q) + \Delta^+(p+l)$$

$$+ \Delta^+(-q-l)|^2_{p_0=E_p, q_0=n_q E_q, l_0=n_l E_l} \sum_{\{n_q, n_l, n_s\}=\pm 1} \delta(E_p + n_q E_q + n_l E_l - n_s E_s)$$

$$\times \mathcal{F}_4(1, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; -n_s, -\mathbf{s}). \tag{112}$$

Except for the matrix element, the right-hand side of this expression is the same as the right-hand side of Eq. (100). Eqs. (100) and (112) have similar structure since both correspond to contributions to the collision term from binary processes. As in the case of Eq. (100), the sum produces three nonzero contributions. The three choices of the n's and the variable transformations that allow us to combine terms are the same as in the ϕ^4 case and are given in Eq. (101). The delta functions and statistical factors are the same as well. A straightforward but tedious calculation shows that the matrix element has the same form for each term. Thus, we finally obtain

$$C_{2L-\phi^3}^{(+)}[p + q \leftrightarrow l + s] = \frac{1}{2!} \int \frac{d^3 q}{(2\pi)^3 E_q} \int \frac{d^3 l}{(2\pi)^3 E_l} \int \frac{d^3 s}{(2\pi)^3 E_s} (2\pi)^4 \delta^{(4)}(\tilde{q} + \tilde{p} - \tilde{l} - \tilde{s})$$

$$\times |\mathcal{M}|^2_{2L-\phi^3}[(1 + f(\mathbf{p}))(1 + f(\mathbf{q}))f(\mathbf{l})f(\mathbf{s}) - f(\mathbf{p})f(\mathbf{q})(1 + f(\mathbf{l}))(1 + f(\mathbf{s}))]. \tag{113}$$

The matrix element squared is given by

$$|\mathcal{M}|^2_{2L-\phi^3} = \frac{g^4}{16} \left| \frac{1}{S} + \frac{1}{T} + \frac{1}{U} \right|^2, \tag{114}$$
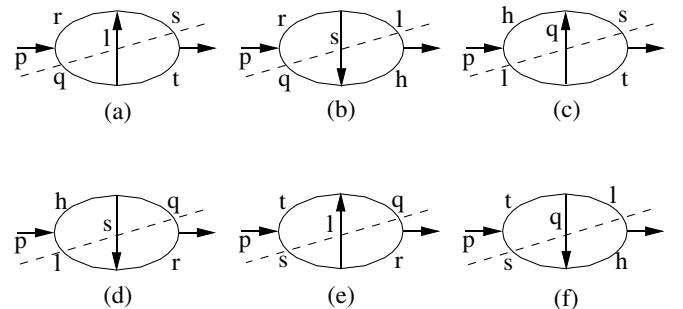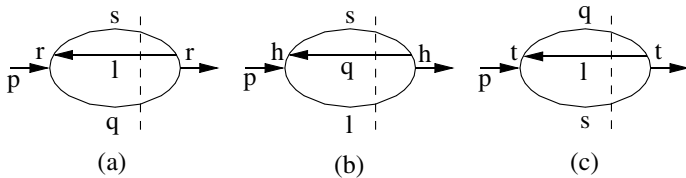

FIG. 11. Three permutations of the centrally cut first diagram in Fig. 8.

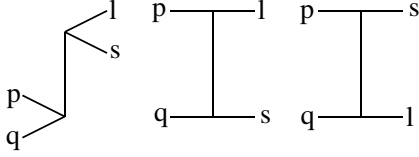FIG. 12. Six permutations of the centrally cut second diagram in Fig. 8.

FIG. 13. Scattering diagrams that contribute to the process $p + q \leftrightarrow l + s$ in ϕ^3 theory.
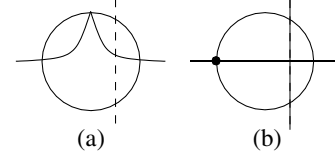

FIG. 14. Three-loop contribution to the ϕ^4 self-energy (a); and an effective two-loop contribution containing a corrected vertex function (b).

where S, T, and U are the Mandelstam variables defined as

$$S \equiv (\tilde{q} + \tilde{p})^2, \qquad T \equiv (\tilde{l} - \tilde{p})^2, \qquad U \equiv (\tilde{q} - \tilde{l})^2. \tag{115}$$

We have used here capital letters to avoid confusion with momentum variables. This matrix element is illustrated in Fig. 13.

XV. THREE-LOOP CONTRIBUTIONS

A. ϕ^4 theory

In ϕ^4 theory, the only three-loop diagram that does not involve tadpoles is shown in Fig. 14(a). This diagram does not have a central cut. The noncentral cut presented in Fig. 14(a) produces a correction to the two-loop sunset graph, as shown in Fig. 14(b), and will not be discussed.

B. ϕ^3 theory

There are diagrams of eight different topologies which contribute to the ϕ^3 self-energy at the three-loop level. They are shown in Fig. 15.

We follow the same strategy as in the previous section and consider only the central cuts of the diagrams in Fig. 15. The internal lines are labeled in all possible ways that produce the same statistical factor. In addition to the definition (109), we define the following momentum variables:

$$\begin{aligned} y &\equiv q + k, & m &\equiv k + l, & j &\equiv k + p, \\ z &\equiv p + q + k, & w &\equiv q + k + l, \\ v &\equiv p + k + l, & u &\equiv p + q + k + l. \end{aligned} \tag{116}$$

In analogy to Eq. (110), we define the operation

$$\ll \cdots \gg_{3L-\phi^3} \stackrel{\text{def}}{=} \Theta(p_0) \frac{g^6 \pi^5}{64} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \sum_{\{n_p, n_q, n_l, n_k, n_u\}=\pm 1} \mathcal{F}_5(n_p, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; n_k, \mathbf{k}; -n_u, -\mathbf{u})$$

$$\times \prod_{\alpha=\{p,q,l,k,u\}} \frac{1}{E_\alpha} \delta(\alpha_0 - n_\alpha E_\alpha) \cdots \tag{117}$$

Combining the results for the diagrams in Figs. 15(a)–15(h), we obtain

$$\begin{aligned} \Theta(p_0) A(p) C_{3L-\phi^3}(p) = \tfrac{1}{24} \ll |&\Delta^+(t)\Delta^-(s) + \Delta^+(y)\Delta^-(z) + \Delta^+(m)\Delta^-(v) + \Delta^+(m)\Delta^-(r) + \Delta^-(r)\Delta^-(z) \\ &+ \Delta^-(r)\Delta^-(s) + \Delta^-(h)\Delta^-(v) + \Delta^+(y)\Delta^-(h) + \Delta^-(h)\Delta^-(s) + \Delta^+(t)\Delta^-(j) \\ &+ \Delta^-(v)\Delta^-(j) + \Delta^-(z)\Delta^-(j) + \Delta^+(m)\Delta^+(w) + \Delta^+(t)\Delta^+(w) \\ &+ \Delta^+(y)\Delta^+(w)|^2 \gg_{3L-\phi^3}. \end{aligned} \tag{118}$$

We can rewrite the result (118) by introducing the delta function $\delta^{(4)}(p + q + k + l - u)$ and an integral over u. As we have done previously, we use Eq. (90) and consider only positive energy contributions to the collision term. The sum over n_q, n_l, n_k, and n_u produces 16 terms. Of these 16 terms, ten are kinematically allowed. Four of these processes can be
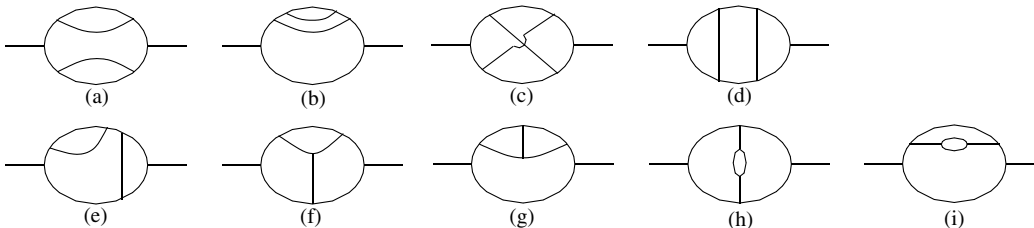

FIG. 15. Three-loop contributions to the ϕ^3 self-energy.

combined by relabeling momentum variables to produce the process symbolically denoted as $p + q \leftrightarrow l + k + u$. The remaining six can be combined to produce the process of the form $p + q + l \leftrightarrow k + u$.

The process $p + q \leftrightarrow l + k + u$

We list the four possible choices of the n's and the variable transformations that allow us to combine terms and obtain the matrix element corresponding to the process $p + q \leftrightarrow l + k + u$:

$$
\begin{aligned}
&[1]\ n_q = -n_l = -n_k = n_u = 1; &&\mathbf{l} \to -\mathbf{l}, &&\mathbf{k} \to -\mathbf{k}; &&q \to \tilde{q}, &&l \to -\tilde{l}, \\
&&&&&&&&k \to -\tilde{k}, &&u \to \tilde{u}; \\
&[2]\ n_q = n_l = -n_k = -n_u = -1; &&\mathbf{q} \to -\mathbf{k}, &&\mathbf{k} \to \mathbf{q}, &&q \to -\tilde{k}, &&l \to -\tilde{l}, \\
&&&\mathbf{l} \to -\mathbf{l}; &&&&k \to \tilde{q}, &&u \to \tilde{u}; \\
&[3]\ n_q = -n_l = n_k = -n_u = -1; &&\mathbf{q} \to -\mathbf{l}, &&\mathbf{l} \to \mathbf{q}, &&q \to -\tilde{l}, &&l \to \tilde{q}, \\
&&&\mathbf{k} \to -\mathbf{k}; &&&&k \to -\tilde{k}, &&u \to \tilde{u}; \\
&[4]\ n_q = n_l = n_k = n_u = -1; &&\mathbf{q} \to -\mathbf{u}, &&\mathbf{l} \to -\mathbf{l}, &&q \to -\tilde{u}, &&k \to -\tilde{k}, \\
&&&\mathbf{k} \to -\mathbf{k}, &&\mathbf{u} \to -\mathbf{q}; &&l \to -\tilde{l}, &&u \to -\tilde{q}.
\end{aligned}
\tag{119}
$$

The delta functions and statistical factor have the same form for each of these four terms. A straightforward but very tedious calculation verifies that the same matrix element is produced in each case. The collision term representing the process $p + q \leftrightarrow l + k + u$ equals

$$
\begin{aligned}
C^{(+)}_{3L-\phi^3}[p + q \leftrightarrow l + k + u] = {} & \frac{1}{3!} \int \frac{d^3q}{(2\pi)^3 E_q} \int \frac{d^3l}{(2\pi)^3 E_l} \int \frac{d^3k}{(2\pi)^3 E_k} \int \frac{d^3u}{(2\pi)^3 E_u} (2\pi)^4 \delta^{(4)}(\tilde{p} + \tilde{q} - \tilde{l} - \tilde{k} - \tilde{u}) \\
& \times |\mathcal{M}|^2_{3L-\phi^3}[p + q \leftrightarrow l + k + u] \\
& \times [(1 + f(\mathbf{p}))(1 + f(\mathbf{q}))f(\mathbf{l})f(\mathbf{k})f(\mathbf{u}) - f(\mathbf{p})f(\mathbf{q})(1 + f(\mathbf{l}))(1 + f(\mathbf{k}))(1 + f(\mathbf{u}))],
\end{aligned}
\tag{120}
$$

where the matrix element squared of the diagrams shown in Fig. 16 is

$$
\begin{aligned}
|\mathcal{M}|^2_{3L-\phi^3}[p + q \leftrightarrow l + k + u] = {} & \frac{g^6}{2^5} |\Delta^-(\tilde{p} + \tilde{q})[\Delta^-(\tilde{k} + \tilde{l}) + \Delta^-(\tilde{k} + \tilde{u}) + \Delta^-(\tilde{l} + \tilde{u})] + \Delta^-(\tilde{p} - \tilde{l})[\Delta^+(\tilde{q} - \tilde{k}) \\
& + \Delta^+(\tilde{q} - \tilde{u}) + \Delta^-(\tilde{k} + \tilde{u})] + \Delta^-(\tilde{p} - \tilde{k})[\Delta^+(\tilde{q} - \tilde{l}) + \Delta^+(\tilde{q} - \tilde{u}) + \Delta^-(\tilde{l} + \tilde{u})] \\
& + \Delta^-(\tilde{p} - \tilde{u})[\Delta^+(\tilde{q} - \tilde{l}) + \Delta^+(\tilde{q} - \tilde{k}) + \Delta^-(\tilde{l} + \tilde{k})] + \Delta^+(\tilde{q} - \tilde{l})\Delta^-(\tilde{k} + \tilde{u}) \\
& + \Delta^+(\tilde{q} - \tilde{k})\Delta^-(\tilde{l} + \tilde{u}) + \Delta^+(\tilde{q} - \tilde{u})\Delta^-(\tilde{l} + \tilde{k})|^2.
\end{aligned}
\tag{121}
$$

The process $q + p + l \leftrightarrow k + u$

We list the six possible choices of the n's and the variable transformations that allow us to combine terms and obtain the matrix element corresponding to the process $p + q + l \leftrightarrow k + u$:

$$
\begin{aligned}
&[1]\ n_q = n_l = -n_k = n_u = 1; &&\mathbf{k} \to -\mathbf{k}; &&k \to -\tilde{k}; \\
&[2]\ n_q = -n_l = n_k = n_u = 1; &&\mathbf{l} \to -\mathbf{k}, &&\mathbf{k} \to \mathbf{l}; &&l \to -\tilde{k}, &&k \to \tilde{l}; \\
&[3]\ n_q = -n_l = -n_k = -n_u = -1; &&\mathbf{p} \to -\mathbf{k}, &&\mathbf{k} \to \mathbf{p}; &&p \to -\tilde{k}, &&k \to \tilde{p}; \\
&[4]\ n_q = -n_l = -n_k = -n_u = 1; &&\mathbf{l} \to -\mathbf{u}, &&\mathbf{k} \to -\mathbf{k}, &&l \to -\tilde{u}, &&k \to -\tilde{k}, \\
&&&\mathbf{u} \to -\mathbf{l}; &&&&u \to -\tilde{l}; \\
&[5]\ n_q = -n_l = n_k = n_u = -1; &&\mathbf{p} \to -\mathbf{u}, &&\mathbf{k} \to -\mathbf{l}, &&p \to -\tilde{u}, &&k \to -\tilde{k}, \\
&&&\mathbf{u} \to -\mathbf{p}; &&&&u \to -\tilde{p}; \\
&[6]\ n_q = n_l = -n_k = n_u = -1; &&\mathbf{p} \to -\mathbf{u}, &&\mathbf{l} \to -\mathbf{k}, &&p \to -\tilde{u}, &&l \to -\tilde{k}, \\
&&&\mathbf{k} \to \mathbf{l}, &&\mathbf{u} \to -\mathbf{p}; &&k \to \tilde{l}, &&u \to -\tilde{p}.
\end{aligned}
\tag{122}
$$

The delta functions and statistical factors have the same form for each of these six terms. Similarly, one can verify that the
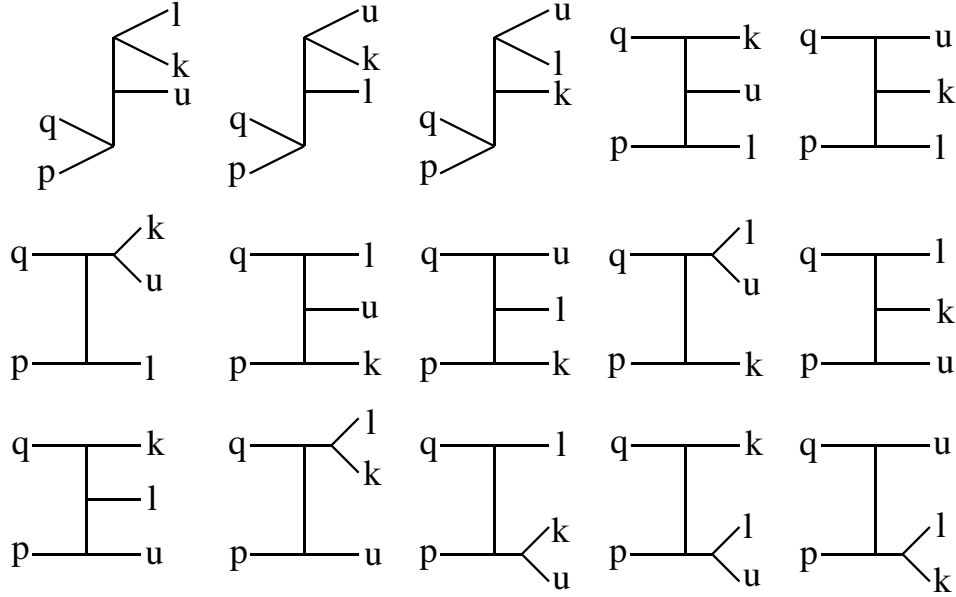
FIG. 16.   Scattering diagrams that contribute to the process $p + q \leftrightarrow l + k + u$ in the order they appear in Eq. (121).

matrix element is the same in each case. The contribution to the collision term from $p + q + l \leftrightarrow k + u$ processes is

$$
C_{3L-\phi^3}^{(+)}[p + q + l \leftrightarrow k + u] = \frac{1}{2!2!} \int \frac{d^3q}{(2\pi)^3 E_q} \int \frac{d^3l}{(2\pi)^3 E_l} \int \frac{d^3k}{(2\pi)^3 E_k} \int \frac{d^3u}{(2\pi)^3 E_u} (2\pi)^4 \delta^{(4)}(\tilde{p} + \tilde{q} - \tilde{l} - \tilde{k} - \tilde{u})
$$
$$
\times |\mathcal{M}|_{3L-\phi^3}^2[p + q + l \leftrightarrow k + u]
$$
$$
\times [(1 + f(\mathbf{p}))(1 + f(\mathbf{q}))(1 + f(\mathbf{l}))f(\mathbf{k})f(\mathbf{u}) - f(\mathbf{p})f(\mathbf{q})f(\mathbf{l})(1 + f(\mathbf{k}))(1 + f(\mathbf{u}))], \quad (123)
$$

where the matrix element squared of the diagrams shown in Fig. 17 equals

$$
|\mathcal{M}|_{3L-\phi^3}^2[p + q + l \leftrightarrow k + u] = \frac{g^6}{2^5} |\Delta^-(\tilde{p} + \tilde{q})[\Delta^-(\tilde{k} - \tilde{l}) + \Delta^-(\tilde{k} + \tilde{u}) + \Delta^-(-\tilde{l} + \tilde{u})] + \Delta^-(\tilde{p} + \tilde{l})[\Delta^+(\tilde{q} - \tilde{k})
$$
$$
+ \Delta^+(\tilde{q} - \tilde{u}) + \Delta^-(\tilde{k} + \tilde{u})] + \Delta^-(\tilde{p} - \tilde{k})[\Delta^+(\tilde{q} + \tilde{l}) + \Delta^+(\tilde{q} - \tilde{u}) + \Delta^-(-\tilde{l} + \tilde{u})]
$$
$$
+ \Delta^-(\tilde{p} - \tilde{u})[\Delta^+(\tilde{q} + \tilde{l}) + \Delta^+(\tilde{q} - \tilde{k}) + \Delta^-(-\tilde{l} + \tilde{k})] + \Delta^+(\tilde{q} + \tilde{l})\Delta^-(\tilde{k} + \tilde{u})
$$
$$
+ \Delta^+(\tilde{q} - \tilde{k})\Delta^-(-\tilde{l} + \tilde{u}) + \Delta^+(\tilde{q} - \tilde{u})\Delta^-(-\tilde{l} + \tilde{k})|^2. \quad (124)
$$

The collision term for ϕ^3 theory at the three-loop level, which represents the $2 \leftrightarrow 3$ processes, is given by the sum of the contributions (120) and (123). Such a form of the collision term was postulated in [9,10] to reproduce the Kubo formula result of the viscosity coefficients within a linearized kinetic theory.

XVI. FOUR-LOOP CONTRIBUTIONS

At the four-loop level, we consider only ϕ^4 theory. The relevant diagrams are the double sunset and two crossed versions of the double sunset, as shown in Fig. 18. All other contributions contain tadpoles and will not be discussed, as has been previously explained.

We follow the strategy formulated in Sec. XIV B and label internal lines of the diagrams in all possible ways that produce the same statistical factor. As previously, we define the operation

$$
\ll \cdots \gg_{4L-\phi^4} \overset{\text{def}}{=} \Theta(p_0) \frac{g^4 \pi^6}{1536} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4m}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4}
$$
$$
\times \sum_{\{n_p, n_q, n_l, n_m, n_k, n_x\} = \pm 1} \mathcal{F}_6(n_p, \mathbf{p}; n_q, \mathbf{q}; n_l, \mathbf{l}; n_m, \mathbf{m}; n_k, \mathbf{k}; -n_x, -\mathbf{x}) \prod_{\alpha = \{p,q,l,m,k,x\}} \frac{1}{E_\alpha} \delta(\alpha_0 - n_\alpha E_\alpha) \cdots \quad (125)
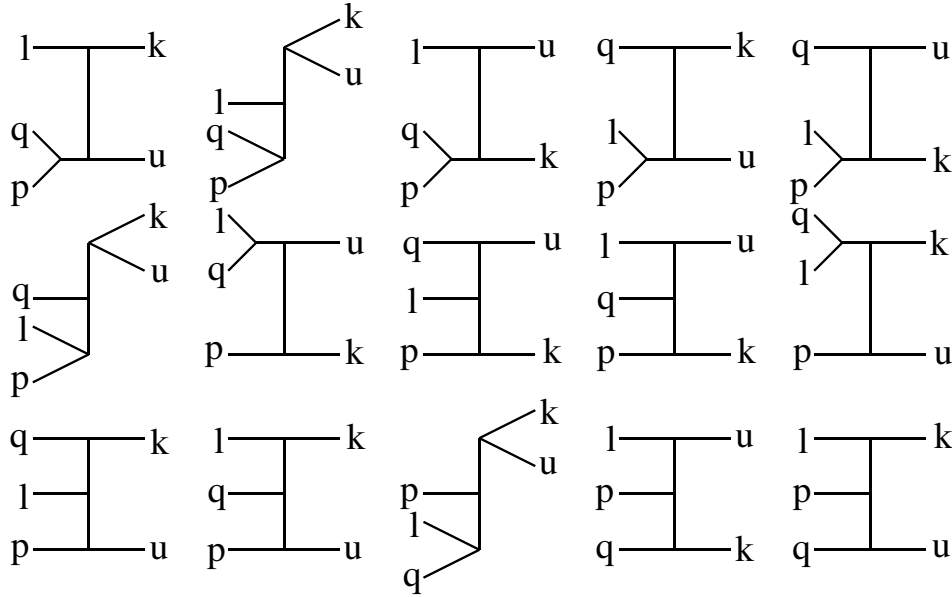$$

FIG. 17.   Scattering diagrams that contribute to the process $p + q + l \leftrightarrow k + u$ in the order they appear in Eq. (124).

where we have used the definitions (109) and (116), and additionally, we have introduced $x \equiv q + p + k + l + m$. After summing up all central cuts of the diagrams shown in Fig. 18, we get the four-loop collision term

$$
\Theta(p_0)A(q)C_{4L-\phi^4}(p) = \frac{1}{10} \ll |\Delta^+(p + q + k) + \Delta^+(p + q + l) + \Delta^+(p + q + m) + \Delta^+(p + k + l)
$$
$$
+ \Delta^+(p + k + m) + \Delta^+(p + l + m) + \Delta^+(k + l + m) + \Delta^+(q + k + l) + \Delta^+(q + l + m)
$$
$$
+ \Delta^+(q + k + m)|^2 \gg_{4L-\phi^4} . \tag{126}
$$

We rewrite this result by introducing the delta function $\delta^{(4)}(q + p + k + l + m - x)$ and an integral over x. As we have done previously, we will use Eq. (90) and consider only positive energy contributions to the collision term. The sum over the remaining n's contains 32 terms. Five of these terms can be combined to give the process $p + q \leftrightarrow k + l + m + x$, ten give the process $p + q + l + m \leftrightarrow k + x$, and ten give the process $p + q + l \leftrightarrow m + k + x$. The remaining seven terms are kinematically forbidden. Below, we give the collision terms representing these processes.

The process $p + q \leftrightarrow k + l + m + x$

$$
C^{(+)}_{4L-\phi^4}[p + q \leftrightarrow l + k + m + x] = \frac{1}{4!} \int \frac{d^3q}{(2\pi)^3 E_q} \int \frac{d^3l}{(2\pi)^3 E_l} \int \frac{d^3k}{(2\pi)^3 E_k} \int \frac{d^3m}{(2\pi)^3 E_m} \int \frac{d^3x}{(2\pi)^3 E_x}
$$
$$
\times (2\pi)^4 \delta^{(4)}(\tilde{p} + \tilde{q} - \tilde{l} - \tilde{k} - \tilde{m} - \tilde{x})|\mathcal{M}|^2_{4L-\phi^4}[p + q \leftrightarrow l + k + m + x]
$$
$$
\times [(1 + f(\mathbf{p}))(1 + f(\mathbf{q}))f(\mathbf{l})f(\mathbf{k})f(\mathbf{m})f(\mathbf{x})
$$
$$
- f(\mathbf{p})f(\mathbf{q})(1 + f(\mathbf{l}))(1 + f(\mathbf{k}))(1 + f(\mathbf{m}))(1 + f(\mathbf{x}))], \tag{127}
$$

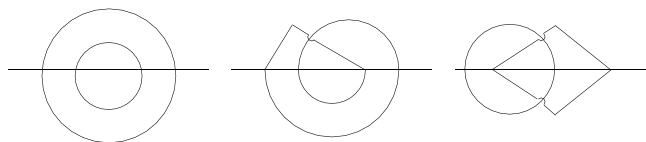where the matrix element squared of the diagrams shown in Fig. 19 equals

FIG. 18.   Four-loop contributions to the ϕ^4 self-energy.

$$|\mathcal{M}|^2_{4L-\phi^4}[p+q \leftrightarrow l+k+m+x] = \frac{g^4}{2^6}|\Delta^-(\tilde{k}+\tilde{l}+\tilde{m}) + \Delta^+(\tilde{q}-\tilde{k}-\tilde{l}) + \Delta^+(\tilde{q}-\tilde{k}-\tilde{m}) + \Delta^+(\tilde{q}-\tilde{l}-\tilde{m})$$
$$+ \Delta^+(\tilde{p}-\tilde{k}-\tilde{l}) + \Delta^+(\tilde{p}-\tilde{k}-\tilde{m}) + \Delta^+(\tilde{p}-\tilde{l}-\tilde{m}) + \Delta^+(\tilde{p}+\tilde{q}-\tilde{k})$$
$$+ \Delta^+(\tilde{p}+\tilde{q}-\tilde{l}) + \Delta^+(\tilde{p}+\tilde{q}-\tilde{m})|^2. \tag{128}$$

The process $p + q + l + m \leftrightarrow k + x$

$$C^{(+)}_{4L-\phi^4}[p+q+l+m \leftrightarrow k+x] = \frac{1}{3!2!} \int \frac{d^3q}{(2\pi)^3 E_q} \int \frac{d^3l}{(2\pi)^3 E_l} \int \frac{d^3m}{(2\pi)^3 E_m} \int \frac{d^3k}{(2\pi)^3 E_k} \int \frac{d^3x}{(2\pi)^3 E_x}$$
$$\times (2\pi)^4 \delta^{(4)}(\tilde{p}+\tilde{q}+\tilde{l}+\tilde{m}-\tilde{k}-\tilde{x})|\mathcal{M}|^2_{4L-\phi^4}[p+q+l+m \leftrightarrow k+x]$$
$$\times [(1+f(\mathbf{p}))(1+f(\mathbf{q}))(1+f(\mathbf{l}))(1+f(\mathbf{m}))f(\mathbf{k})f(\mathbf{x})$$
$$- f(\mathbf{p})f(\mathbf{q})f(\mathbf{l})f(\mathbf{m})(1+f(\mathbf{k}))(1+f(\mathbf{x}))], \tag{129}$$

where the matrix element squared of the diagrams shown in Fig. 20 equals

$$|\mathcal{M}|^2_{4L-\phi^4}[p+q+l+m \leftrightarrow k+x] = \frac{g^4}{2^6}|\Delta^+(\tilde{l}+\tilde{m}-\tilde{k}) + \Delta^+(\tilde{q}-\tilde{k}+\tilde{l}) + \Delta^+(\tilde{q}-\tilde{k}+\tilde{m}) + \Delta^+(\tilde{q}+\tilde{l}+\tilde{m})$$
$$+ \Delta^+(\tilde{p}-\tilde{k}+\tilde{l}) + \Delta^+(\tilde{p}-\tilde{k}+\tilde{m}) + \Delta^+(\tilde{p}+\tilde{l}+\tilde{m})$$
$$+ \Delta^+(\tilde{p}+\tilde{q}-\tilde{k}) + \Delta^+(\tilde{p}+\tilde{q}+\tilde{l}) + \Delta^+(\tilde{p}+\tilde{q}+\tilde{m})|^2. \tag{130}$$

The process $p + q + l \leftrightarrow m + k + x$

$$C^{(+)}_{4L-\phi^4}[p+q+l \leftrightarrow m+k+x] = \frac{1}{2!3!} \int \frac{d^3q}{(2\pi)^3 E_q} \int \frac{d^3l}{(2\pi)^3 E_l} \int \frac{d^3m}{(2\pi)^3 E_m} \int \frac{d^3k}{(2\pi)^3 E_k} \int \frac{d^3x}{(2\pi)^3 E_x}$$
$$\times (2\pi)^4 \delta^{(4)}(\tilde{p}+\tilde{q}+\tilde{l}-\tilde{m}-\tilde{k}-\tilde{x})|\mathcal{M}|^2_{4L-\phi^4}[p+q+l \leftrightarrow m+k+x]$$
$$\times [(1+f(\mathbf{p}))(1+f(\mathbf{q}))(1+f(\mathbf{l}))f(\mathbf{m})f(\mathbf{k})f(\mathbf{x})$$
$$- f(\mathbf{p})f(\mathbf{q})f(\mathbf{l})(1+f(\mathbf{m}))(1+f(\mathbf{k}))(1+f(\mathbf{x}))], \tag{131}$$

where the matrix element squared of the diagrams shown in Fig. 21 equals

$$|\mathcal{M}|^2_{4L-\phi^4}[p+q+l \leftrightarrow m+k+x] = \frac{g^4}{2^6}|\Delta^+(\tilde{l}-\tilde{k}-\tilde{m}) + \Delta^+(\tilde{q}-\tilde{k}+\tilde{l}) + \Delta^+(\tilde{q}-\tilde{k}-\tilde{m}) + \Delta^+(\tilde{q}+\tilde{l}-\tilde{m})$$
$$+ \Delta^+(\tilde{p}-\tilde{k}+\tilde{l}) + \Delta^+(\tilde{p}-\tilde{k}-\tilde{m}) + \Delta^+(\tilde{p}+\tilde{l}-\tilde{m}) + \Delta^+(\tilde{p}+\tilde{q}-\tilde{k})$$
$$+ \Delta^+(\tilde{p}+\tilde{q}+\tilde{l}) + \Delta^+(\tilde{p}+\tilde{q}-\tilde{m})|^2. \tag{132}$$
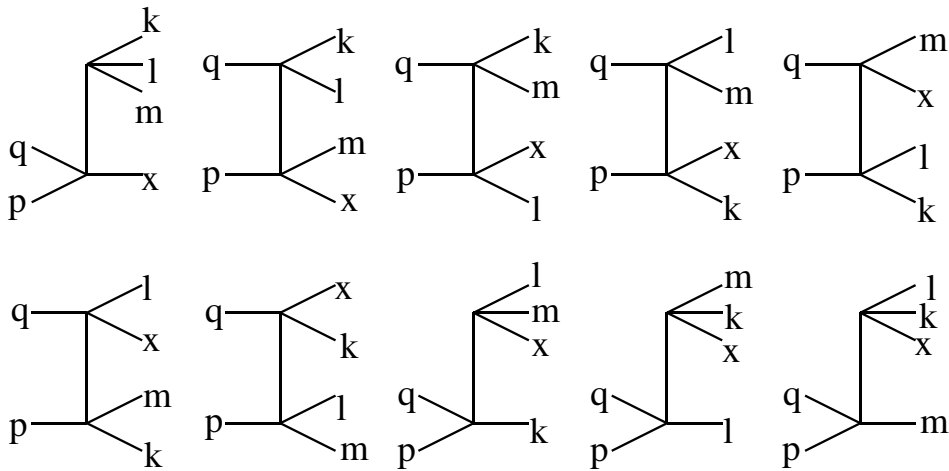

FIG. 19.   Scattering diagrams that contribute to the process $p + q \leftrightarrow k + l + m + x$ in the order they appear in Eq. (128).
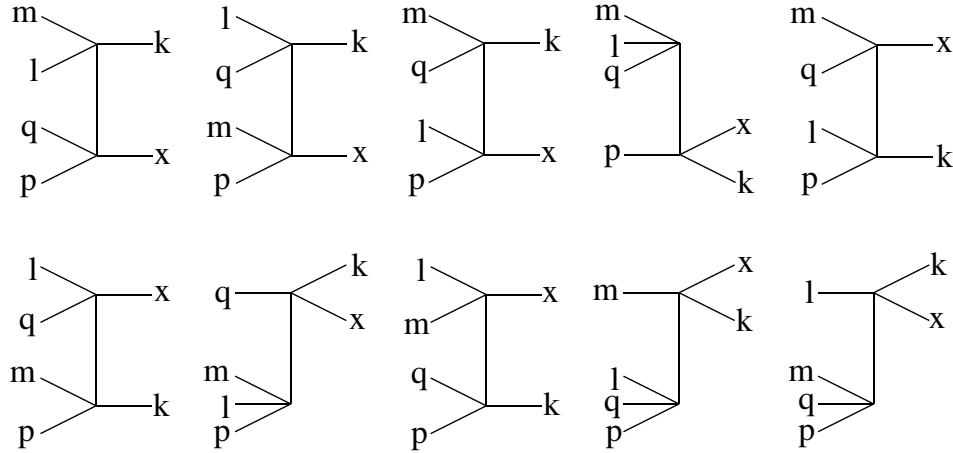
FIG. 20. Scattering diagrams that contribute to the process $p + q + l + m \leftrightarrow k + x$ in the order they appear in Eq. (130).

The collision term of the ϕ^4 theory at four-loop level, which represents the $2 \leftrightarrow 4$ and $3 \leftrightarrow 3$ processes, is given by the sum of the contributions (127), (129), and (131). As in the case of ϕ^3 interaction, a form of the collision term beyond binary approximation was postulated in [9,10] to reproduce the Kubo formula result of the viscosity coefficients within a linearized kinetic theory. While the contributions representing the $2 \leftrightarrow 4$ processes coincide, the term corresponding to the $3 \leftrightarrow 3$ interactions is absent in the effective transport equation [9,10], as it is supposed to give negligible contribution to the viscosity [41]. As already mentioned, the scattering $3 \leftrightarrow 3$ was discussed in [8].

In performing the perturbative calculations of the collision terms in Secs. XIII, XIV, XV, and XVI, we have used noninteracting Green's functions. However, the transport equation is of interest precisely because its solution is the full interacting distribution function. The terms on the left-hand side of the transport equation clearly depend on this interacting distribution function. On the right-hand side, we use the perturbatively calculated collision term and replace the noninteracting distribution function by the interacting one. We drop gradient terms in the expansion of the interacting distribution function about the noninteracting distribution, when these gradient terms appear within the collision term. The justification for this procedure is as follows: we consider the gradient and coupling constant approximations as independent, and choose to investigate higher order terms in the coupling constant approximation, while working at lowest order in the gradient approximation. The contributions we have calculated correspond physically to multiparticle production processes involving zero-width quasiparticles. We remind the reader that finite-width quasiparticles will give rise to contributions to the collision term that are kinematically forbidden for zero-width quasiparticles. For ϕ^4 theory an example is a $1 \leftrightarrow 3$ process. In this paper we assume infinitely narrow quasiparticles [Eq. (29)], and in this limit the effects of these processes will be negligible.

XVII. SUMMARY AND OUTLOOK

Kinetic equations usually take into account the interaction of particles with a mean-field, and interparticle pair collisions. However, when the system of interest is very dense, multiparticle interactions are expected to play a significant role in the system's dynamics. In addition, if a characteristic particle's kinetic energy is comparable to the particle's mass, particle production processes become important. Both multiparticle interactions and production processes occur in systems of relativistic quantum fields, such a quark-gluon plasma, when the energy density is sufficiently high.

In this study, we have given the first systematic derivation of the transport equation of relativistic quantum fields which takes into account multiparticle and production process. Using the Schwinger-Keldysh approach, we have discussed scalar fields with cubic and quartic interactions. Mean-field phenomena are controlled by the one-loop tadpole contributions to the self-energy, and binary collisions correspond to the two-loop graphs. Multiparticle process occur at the three-loop level in ϕ^3 theory and at the four-loop level in ϕ^4 theory. Analysis of three- and four-loop diagrams is a technically very complex problem which requires special computational techniques. We have used the Keldysh representation because it provides a clean separation between propagators that correspond to real and virtual particles. In addition, we have made use of the cut structure of the collision term. Our calculations have been performed with the help of a program that uses MATHEMATICA symbolic manipulation software.

Throughout this paper, we have assumed that quasiparticles are of zero-width and that their four-momenta are on the mass shell. This assumption is crucial since some processes which occur at one- and two-loop level, like $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ processes, are kinematically forbidden for on-mass-shell particles. We have derived the explicit form of the contributions to the collision term of the transport equation corresponding to the processes: $2 \leftrightarrow 3$ for the ϕ^3 model, and $2 \leftrightarrow 4$ and $3 \leftrightarrow 3$ for the ϕ^4 model.
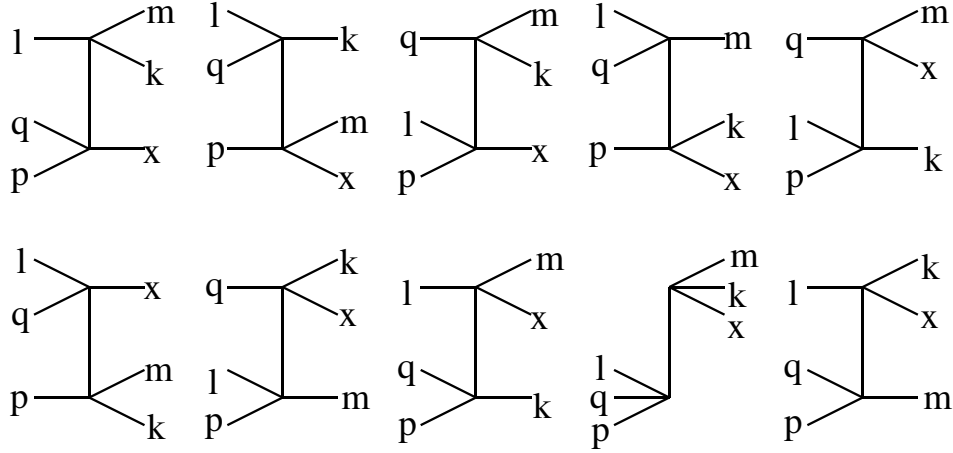
FIG. 21.   Scattering diagrams that contribute to the process $p + q + l \leftrightarrow m + k + x$ in the order they appear in Eq. (132).

In this paragraph we clearly state the domain of applicability of our final results. Our derivation relies on several assumptions and approximations which are discussed in Sec. V. First of all, the system has to be homogeneous at a scale which is smaller than or comparable to the characteristic inverse momentum and the inverse effective mass, as required by the conditions (26) and (27). The system has to be weakly interacting so that the loop expansion is a legitimate approximation. This requires smallness of the coupling constants. Since we consider quasiparticles which are on the mass shell, the conditions (27) and (29) have to be fulfilled. The latter is easily satisfied if the bare mass is much larger than the medium correction to the mass. A system of weakly interacting massive scalar fields which is close to equilibrium satisfies all of these conditions.

In the future we plan to study further the multiparticle collision terms derived here. Although the integrals seem to be regular for massive particles, they require very careful analysis, as the transport theory of dense gases or liquids is known to suffer serious problems at the level of multiparticle interactions, see, e.g., [42]. The integral representing the process $3 \leftrightarrow 3$ is of particular concern here as one particle can experience zero momentum transfer producing a singularity of the respective propagator.

We intend to extend the work of Refs. [9,10] on the role of multiparticle processes in transport phenomena. However, we are going to start with the transport equations, which are derived here, and not with the Kubo formulas. We also plan to study how the system reaches chemical equilibrium due to particle production processes. The first step in this direction is to compute the interaction rate

$$\Gamma[n \to m] \stackrel{\text{def}}{=} \frac{1}{n} \int \frac{d^3 p}{(2\pi)^3 E_p} C^{(+)}[p + p_1 + p_2$$
$$+ \cdots p_{n-1} \to q_1 + q_2 + \cdots + q_m], \quad (133)$$

where $C^{(+)}[\cdots]$ is the gain contribution to the collision term corresponding to the process $p + p_1 + p_2 + \cdots p_{n-1} \to q_1 + q_2 + \cdots + q_m$. Relative rates of the form $\Gamma[2 \to 3]/\Gamma[2 \to 2]$ or $\Gamma[3 \to 3]/\Gamma[2 \to 2]$ will give a measure of the importance of multiparticle processes.

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