# Color Instabilities of the Quark-Gluon Plasma

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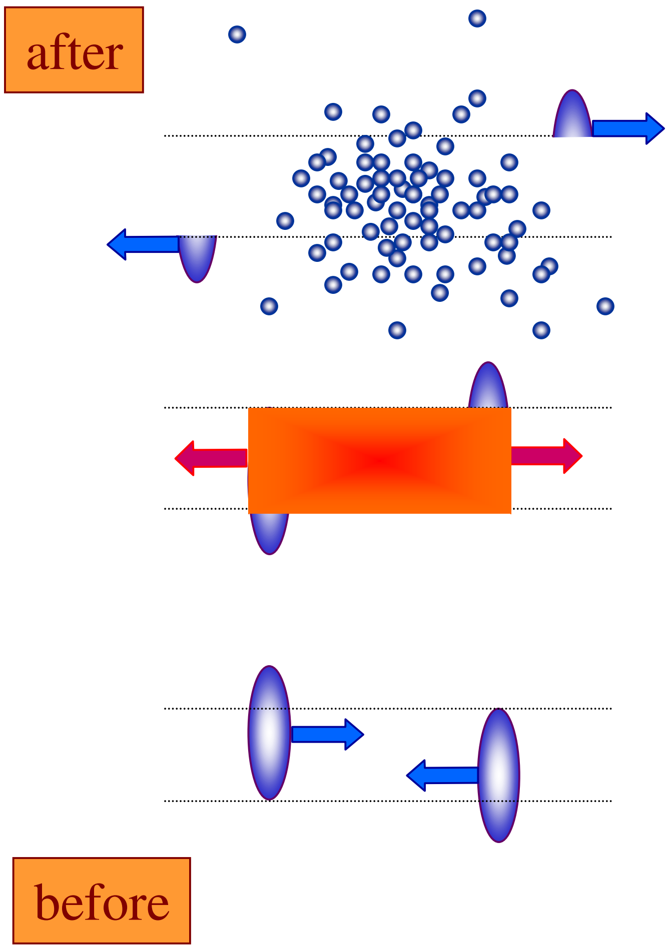
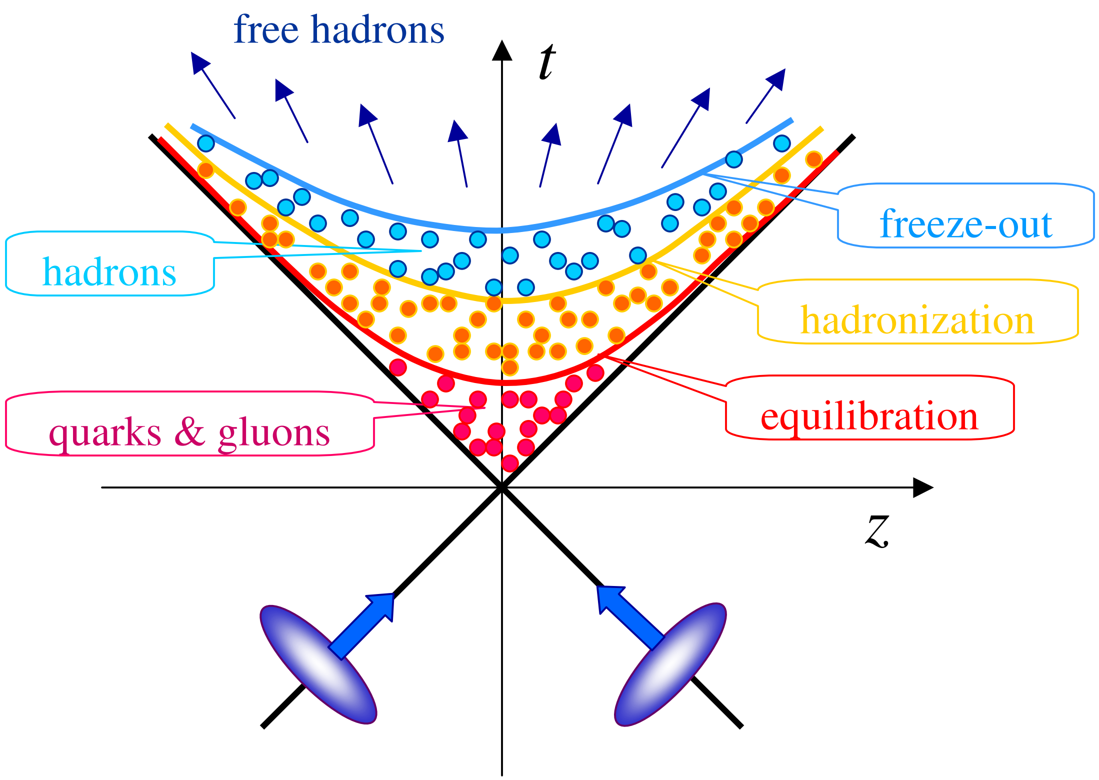
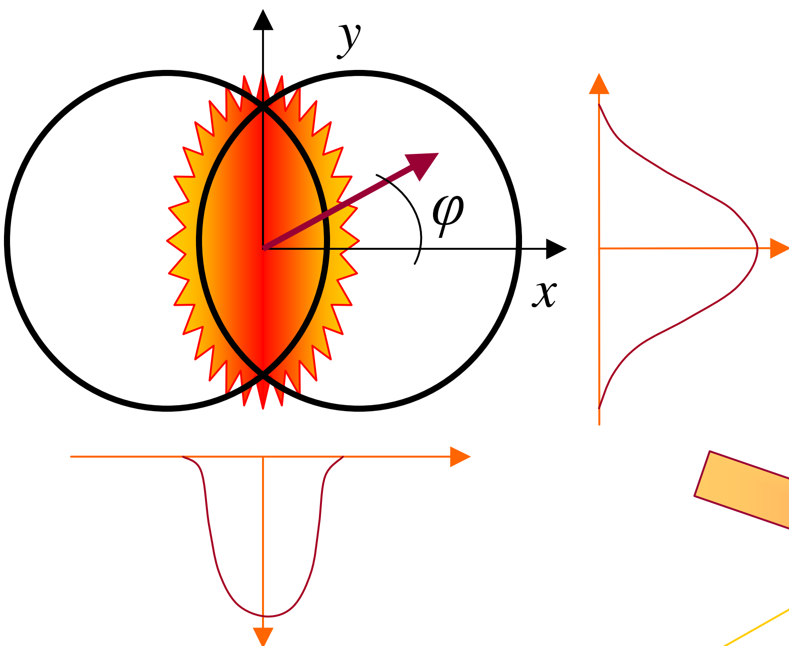
# Course of relativistic heavy-ion collisions



# Evidence of the early stage equilibration

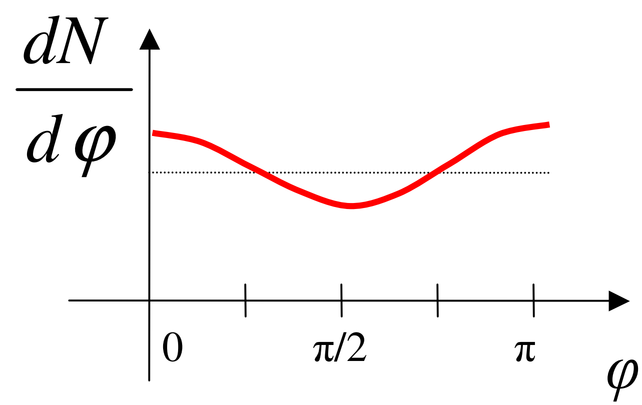
**Success of hydrodynamic models in describing elliptic flow**




Hydrodynamics

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\mathbf{v} = -\frac{\nabla p}{\rho}$$

Hydrodynamics requires local thermodynamical equilibrium!



## Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

**Eccentricity decays due to the free streaming!**

$\varepsilon \searrow \;\Rightarrow\; v_2 \searrow$ ➡ $t_{eq} \leq 1 \ \mathrm{fm}/c$

time of equilibration

U. Heinz, AIP Conf. Proc.**739,** 163 (2004)

# Collisions in weakly coupled QGP

Assumption: **QGP is weakly coupled !** $\alpha_s \equiv \frac{g^2}{4\pi} << 1$ – QCD coupling constant

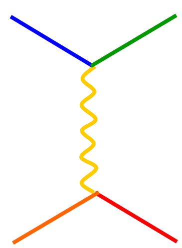
**Time scale of equilibration driven by hard parton-parton scatterings**

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

hard scattering ~ momentum transfer of order of $T$

either single hard scattering or multiple soft scatterings

dominated by



**Collisions are too slow !**

# Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

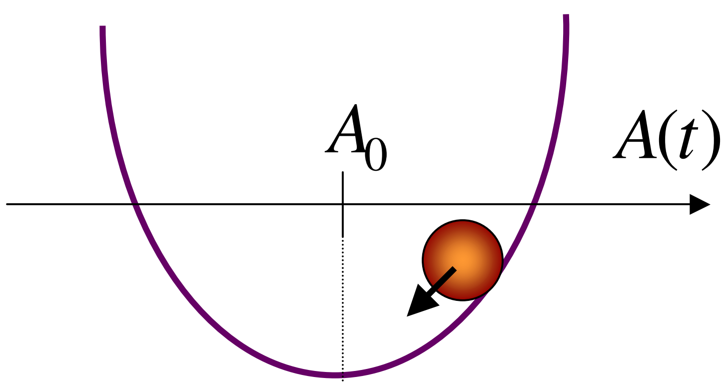
fluctuation

**Instability**

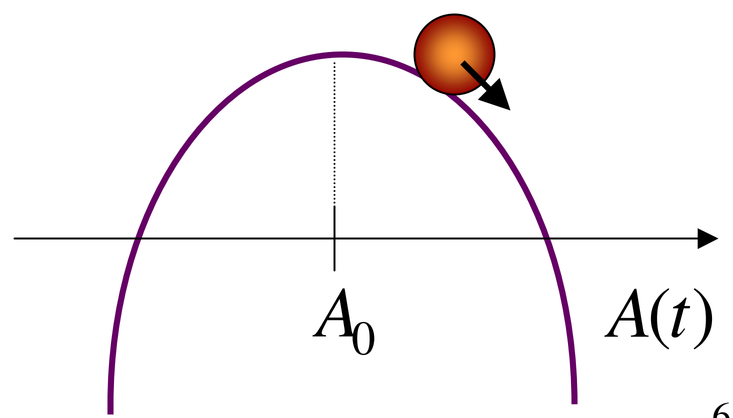
$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

**stable configuration**




**unstable configuration**



# Plasma instabilities

- **instabilities in configuration space – hydrodynamic instabilities**

- **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

$$f(\mathbf{p}) \text{ is not } \sim \exp\left(-\frac{E}{T}\right)$$

# Kinetic instabilities

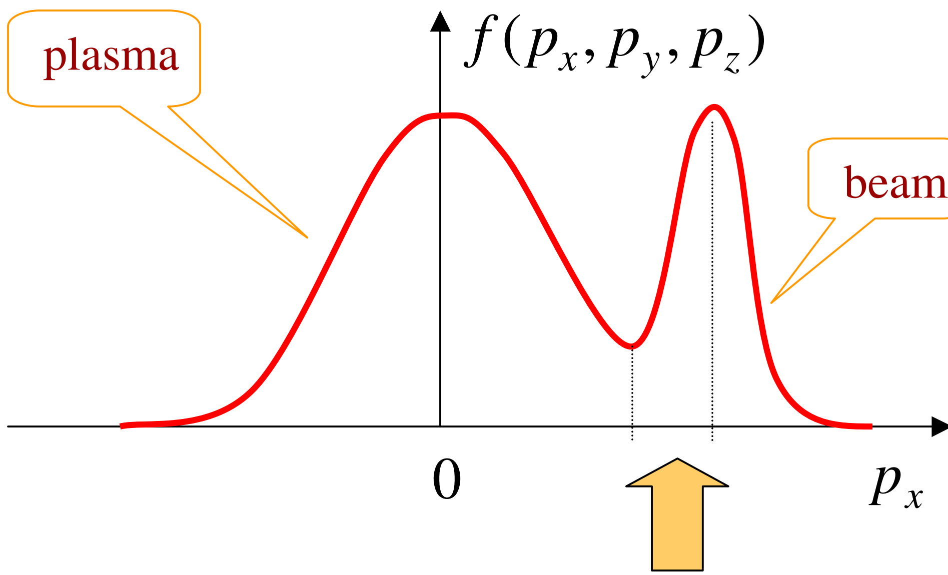
- **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}, \;\; \delta\rho \sim e^{-i(\omega t - \mathbf{kr})}$

- **transverse modes** – $\mathbf{k} \perp \mathbf{E}, \;\; \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{kr})}$

**E** – electric field, **k** – wave vector, ρ – charge density, **j** - current

# Logitudinal modes

**unstable configuration**

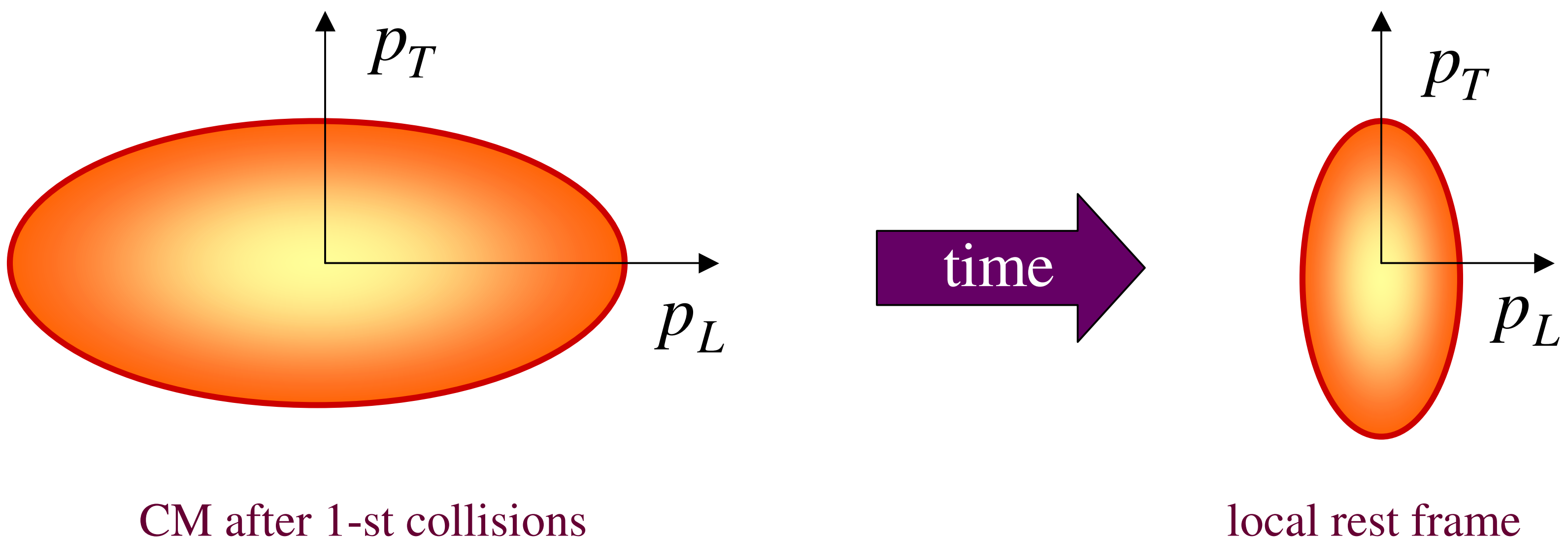



**Energy is transferred from particles to fields**

# Transverse modes

**Unstable modes occur due to anisotropy of the momentum distribution**

**Parton momentum distribution is initially strongly anisotropic**



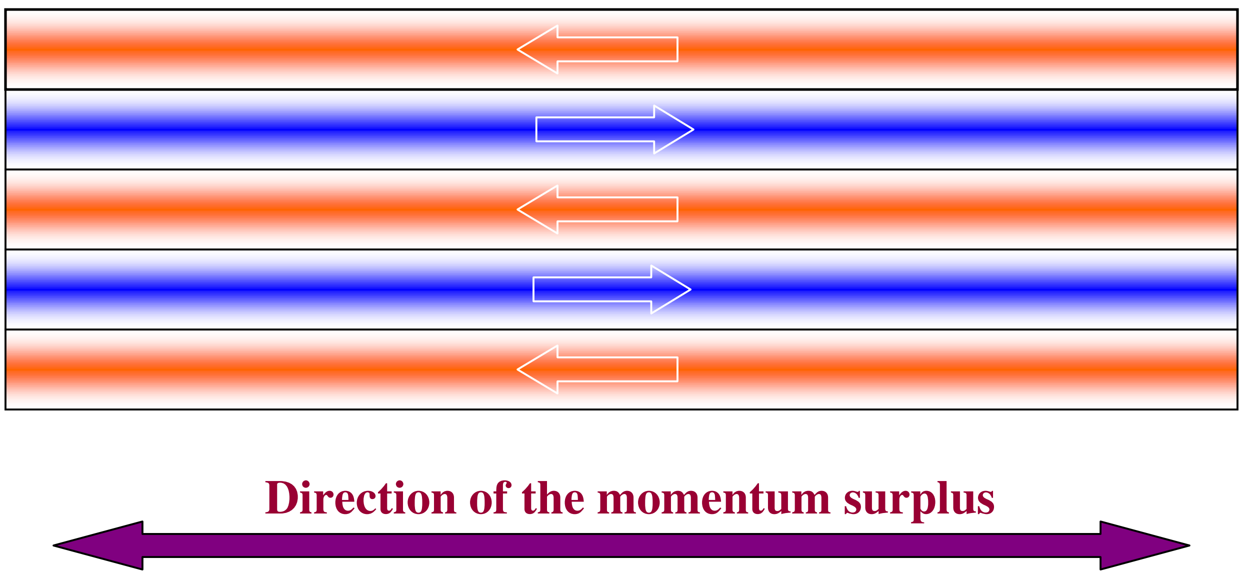

**Transverse modes are relevant for relativistic nuclear collisions!**

# Seeds of instability

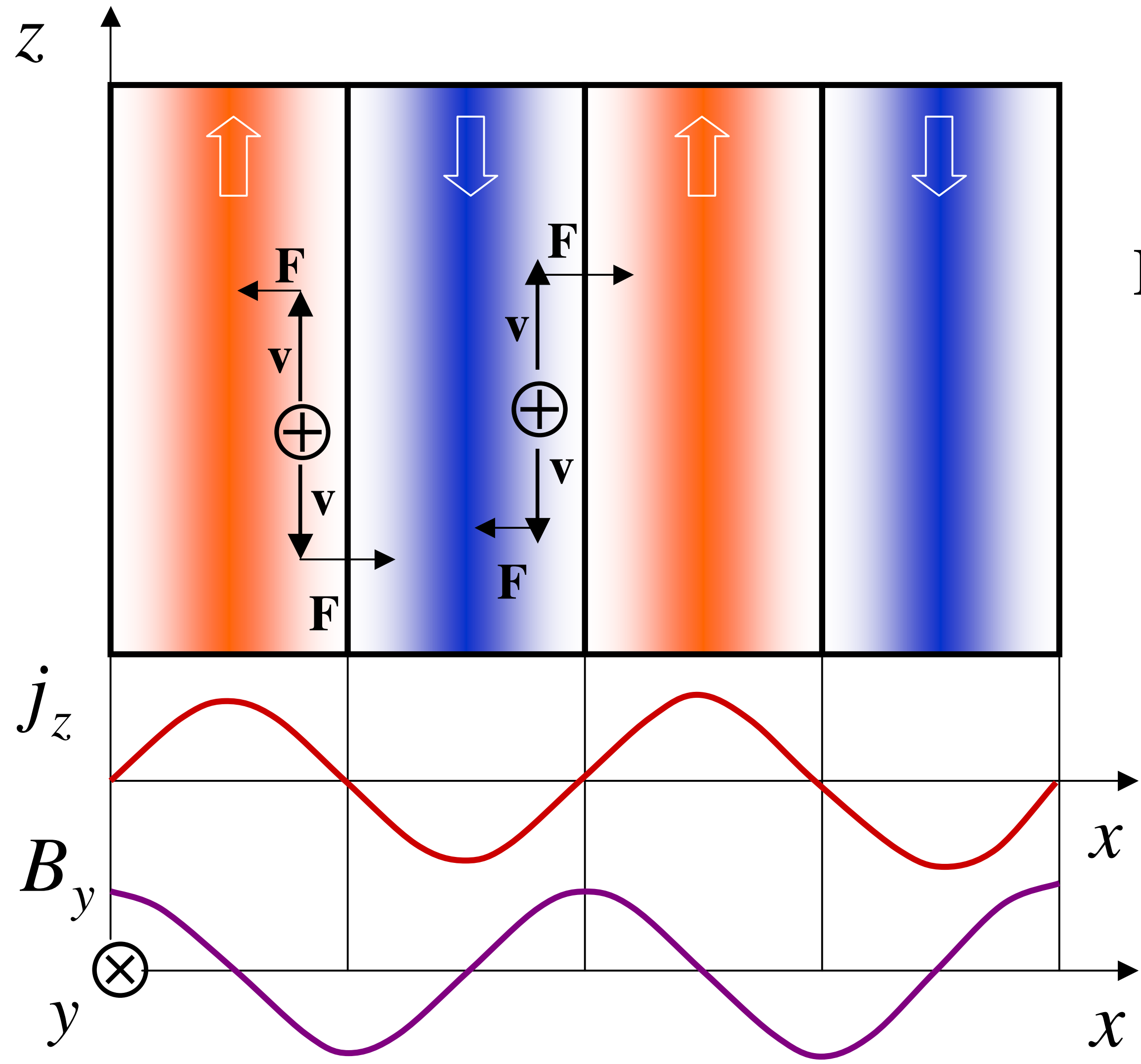
$\left\langle j_a^{\mu}(x) \right\rangle = 0$ **but current fluctuations are finite**

$$\left\langle j_a^{\mu}(x_1)\, j_b^{\nu}(x_2) \right\rangle = \frac{1}{2}\delta^{ab} \int \frac{d^3p}{(2\pi)^3} \frac{p^{\mu} p^{\nu}}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0$$

$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



# Mechanism of filamentation




Lorentz force

$$\mathbf{F} = q\,\mathbf{v}\times\mathbf{B}$$

Ampere's law

$$\nabla\times\mathbf{B} = \mathbf{j}$$

# Dispersion equation

**Equation of motion of chromodynamic field $A^\mu$ in momentum space**

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

**Dispersion equation**

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

**Instabilities – solutions with Im$\omega$ > 0** $\Rightarrow$ $A^\mu(x) \sim e^{\mathrm{Im}\,\omega t}$

**Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. How to get it?**

# Transport theory

fundamental:

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu Q\} = C$$ quarks

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu \overline{Q}\} = \overline{C}$$ antiquarks

adjoint:

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu\nu}, (x)\partial_p^\nu G\} = C_g$$ gluons

free streaming | mean-field force | collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ....], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]$$ mean-field generation

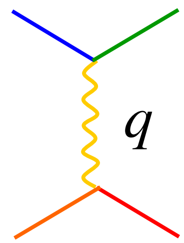
collisionless limit: $C = \overline{C} = C_g = 0$

# Time scale of collisional processes

Time scale of processes driven by parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

$$t_{\text{soft}} \sim \frac{1}{g^2 \ln(1/g)\, T}$$




hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{\text{collec}} \sim \frac{1}{g\, T}$$

$$g^2 \ll 1 \quad \Rightarrow \quad t_{\text{hard}} \gg t_{\text{soft}} \gg t_{\text{collec}}$$

The instabilities are fast!

# Transport theory - linearization

fluctuation

$$Q(p,x) = Q_0(p) + \delta Q(p,x)$$

stationary colorless state $Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| >> |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p)| >> |\partial_p^\mu \delta Q(p,x)|$$

Linearized transport equations

$$p_\mu D^\mu \delta Q(p,x) - g p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p) = 0$$

$$p_\mu D^\mu \delta \overline{Q}(p,x) + g p^\mu F_{\mu\nu}(x) \partial_p^\nu \overline{Q}_0(p) = 0$$

$$p_\mu \mathcal{D}^\mu \delta G(p,x) - g p^\mu \mathcal{F}_{\mu\nu}(x) \partial_p^\nu G_0(p) = 0$$

# Transport theory – polarization tensor

$$\delta Q(p,x) = g\int d^4x'\,\Delta_p(x-x')\,p^\mu F_{\mu\nu}(x)\partial_p^\nu Q_0(p)$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu[\delta Q, \delta\overline{Q}, \delta G]$$

$$j^\mu(k) = \Pi^{\mu\nu}(k)A_\nu(k)$$

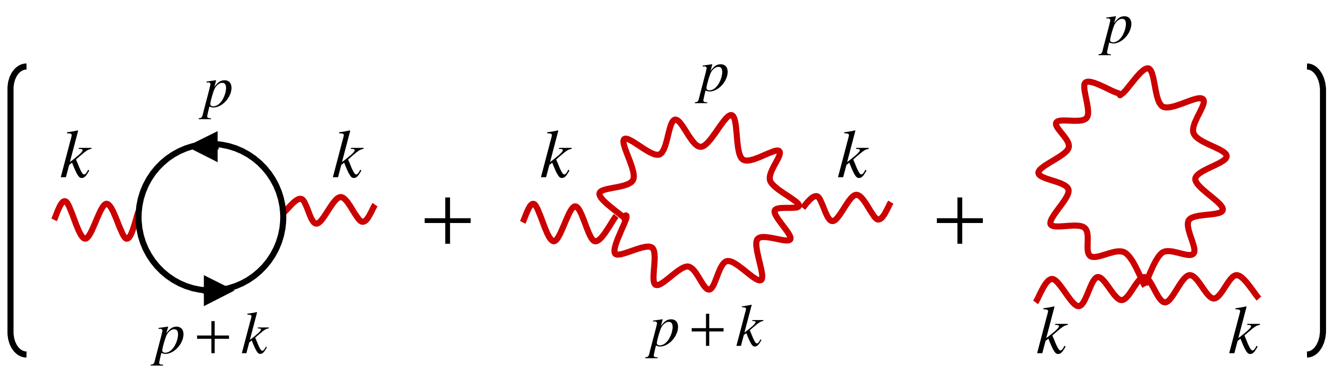
$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \bar{n}(\mathbf{p}) + 2n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\Big[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\Big]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

# Diagrammatic Hard Loop approach

$\Pi^{\mu\nu}(k) =$




Hard loop approximation: $k^\mu << p^\mu$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\left[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\right]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$



St. M. & M. Thoma, Phys. Rev. C **62**, 036011 (2000)

# Dispersion equation

**Dispersion equation**

$$\det[k^2 g^{\mu\nu} - k^{\mu}k^{\nu} - \Pi^{\mu\nu}(k)] = 0$$

$$k_{\mu}\Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$ **chromodielectric tensor**

$$k^{\mu} \equiv (\omega, \mathbf{k})$$

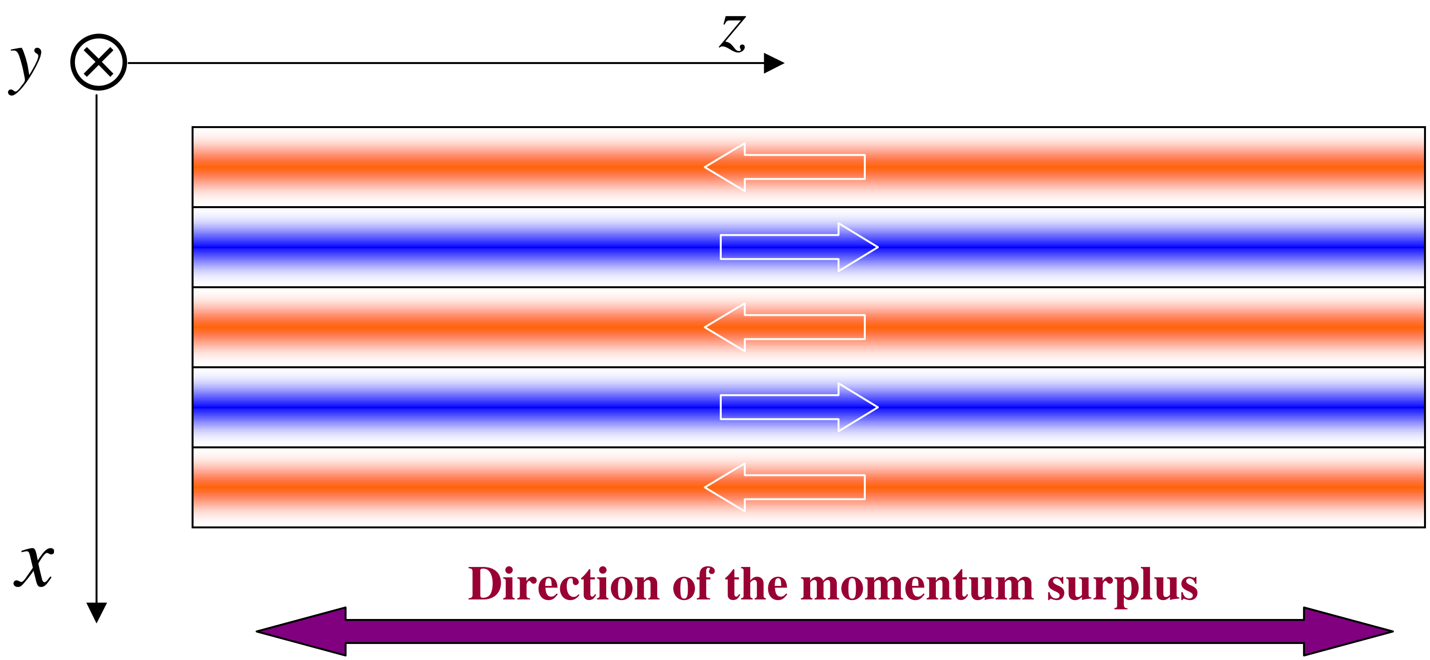
**Dispersion equation**

$$\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int \frac{d^3p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{kv} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l}\Big[\Big(1 - \frac{\mathbf{kv}}{\omega}\Big)\delta^{lj} + \frac{k^l v^j}{\omega}\Big]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

# Dispersion equation – configuration of interest




$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

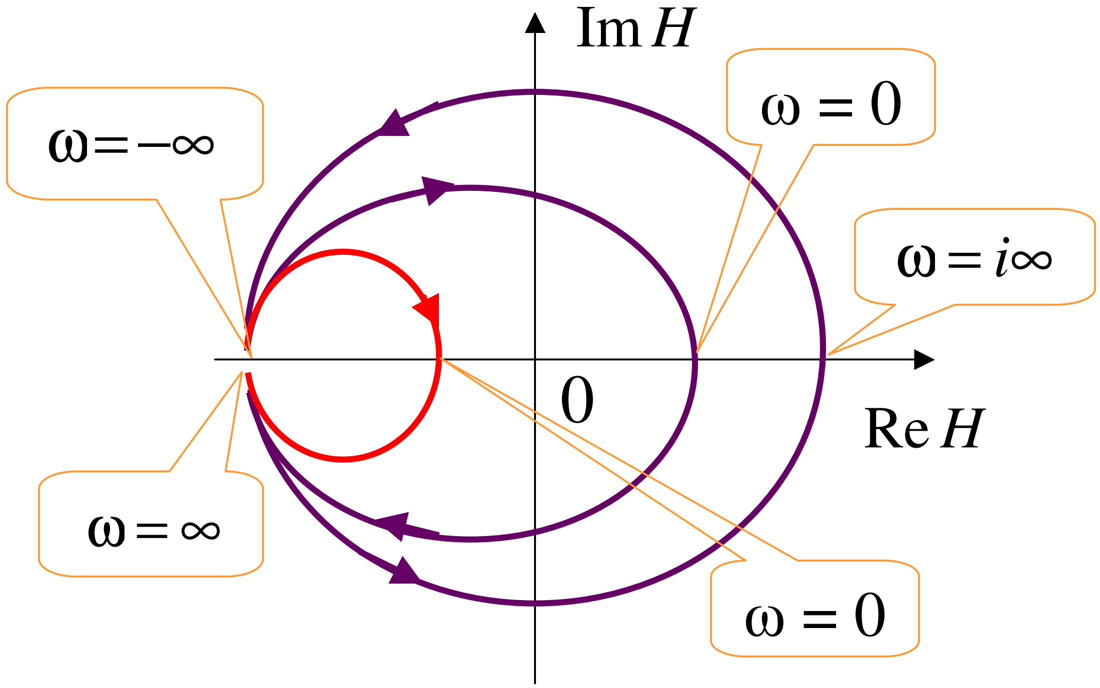
**Dispersion equation**

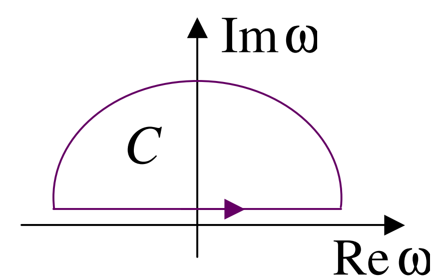
$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

# Existence of unstable modes – Penrose criterion

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$




**There are unstable modes if**

$$H(\omega = 0) < 0$$

**Anisotropy!**

# Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho\sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2+\sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$$\rho = 6\ \mathrm{fm}^{-3}$$

$$\alpha_s = g^2/4\pi = 0.3$$

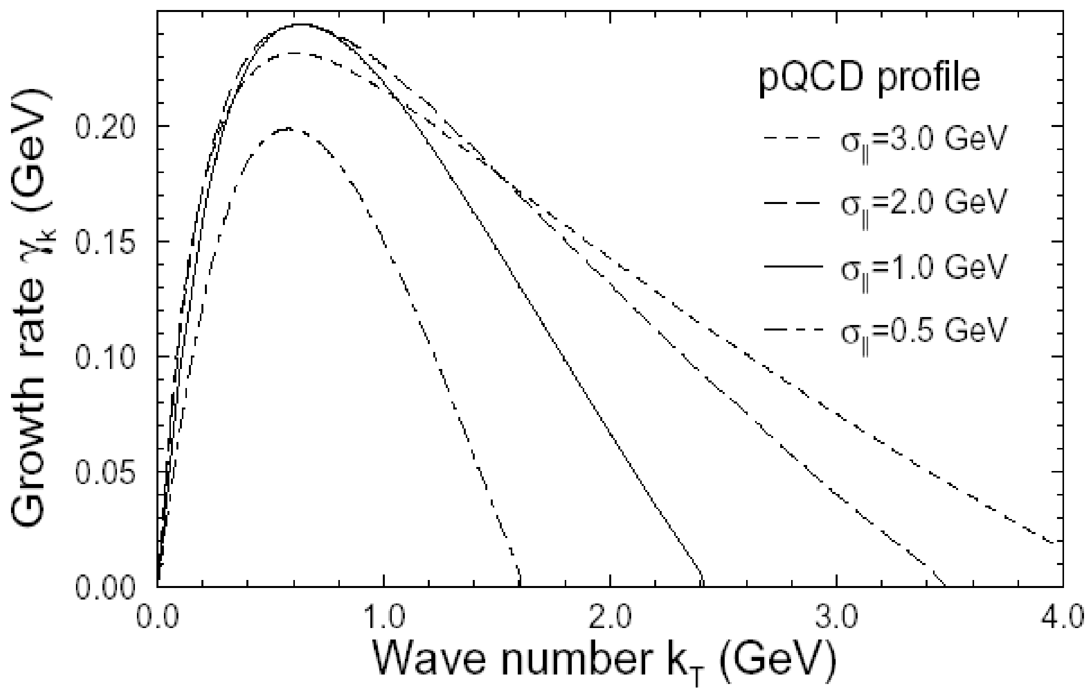
$$\sigma_\perp = 0.3\ \mathrm{GeV}$$

$$k^2 - \omega^2\varepsilon^{zz}(\omega,k) = 0$$

**solution**

$$\omega(k) = \pm i\,\gamma_k$$

$$0 < \gamma_k \in \Re$$




J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

# Hard-Loop dynamics

**Soft fields in the passive background of hard particles**

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\rm eff} = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\Big[ f(\mathbf{p}) F^a_{\mu\nu}(x)\left(\frac{p^\nu p^\rho}{(p\cdot D)^2}\right)_{ab} F^{b\mu}_{\rho}(x)$$

$$+ i\,\frac{C_F}{3}\,\tilde{f}(\mathbf{p})\,\bar\psi(x)\,\frac{p\cdot\gamma}{p\cdot D}\,\psi(x)\Big]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

St. M., A. Rebhan & M. Strickland, Phys. Rev. D **74**, 025004 (2004)

# Growth of instabilities – 1+1 numerical simulations

**SU(2) Hard Loop Dynamics**

**1+1 dimensions**

$$A_a^\mu = A_a^\mu(t, z)$$

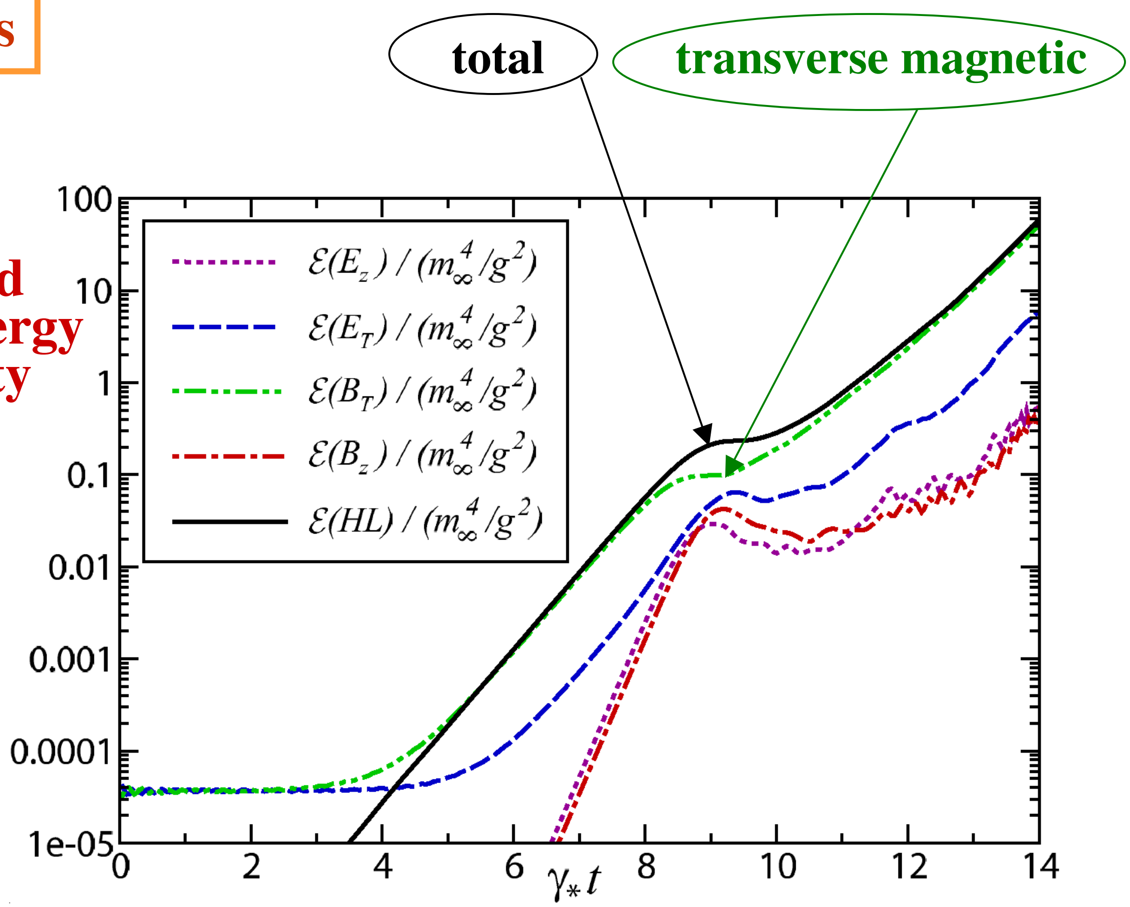
**Anisotropic particle's momentum distribution**

$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi} \int_0^\infty dp\, p^2 \frac{df_{\text{iso}}(p)}{dp}$$

$$(m_D, \zeta)$$

**Scaled field energy density**

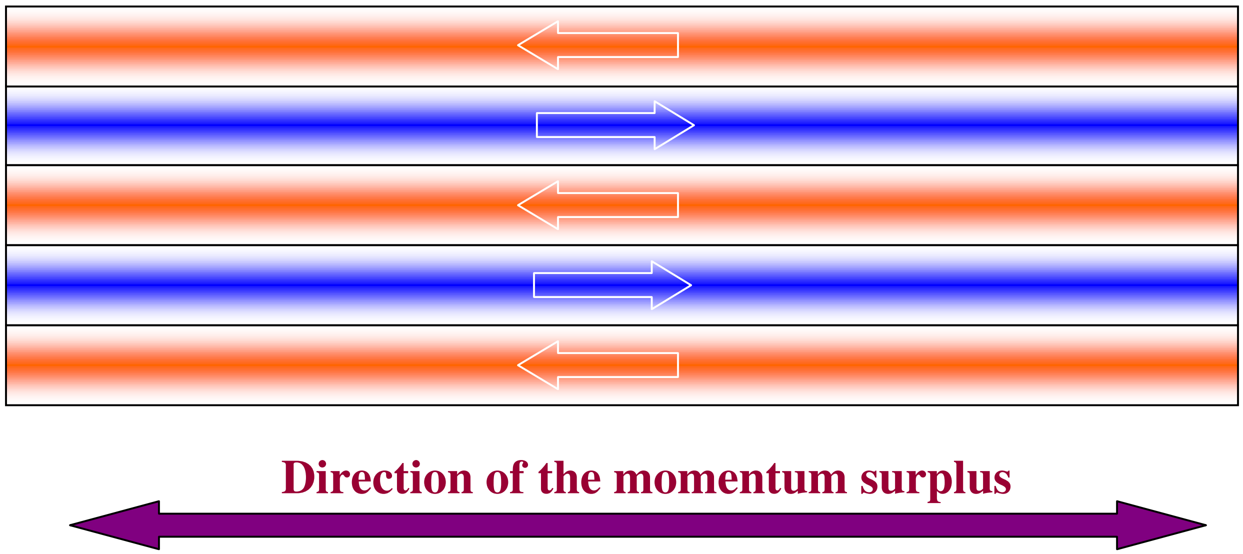


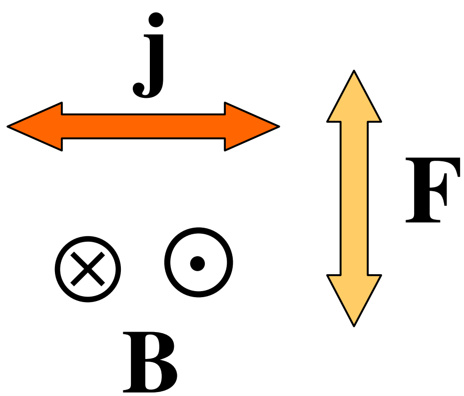

Strong anisotropy $\zeta = 10$

$\gamma_*$ - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

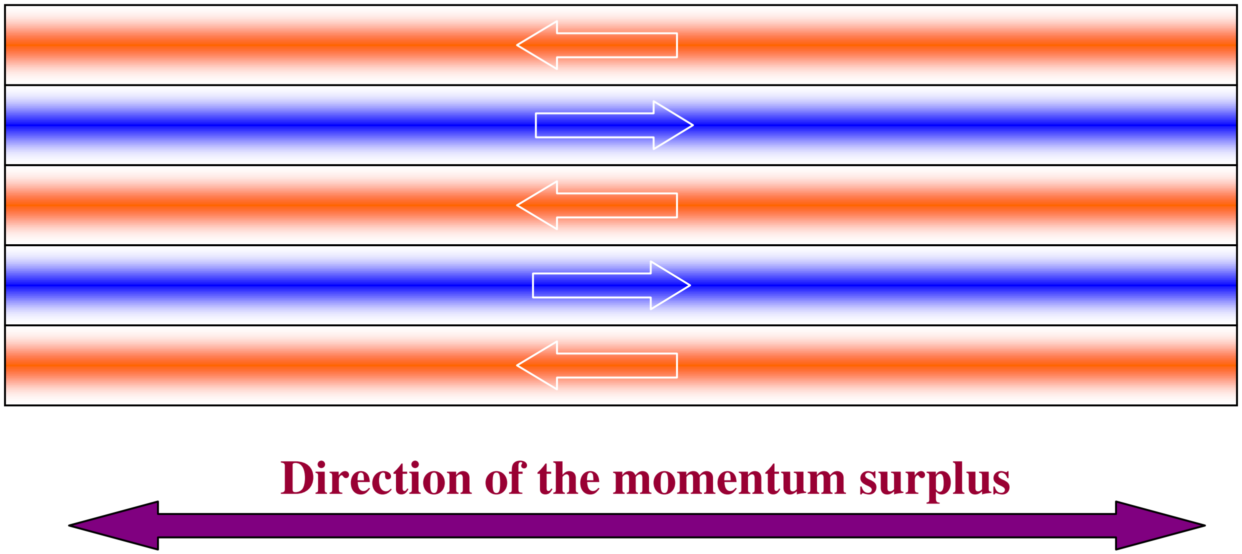
# Isotropization - particles

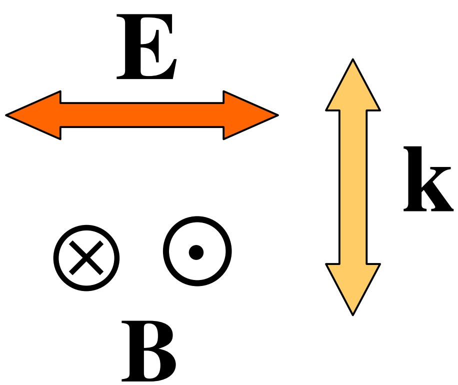



$$\Delta \mathbf{p} = \int dt \, \mathbf{F}$$

# Isotropization - fields




$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$

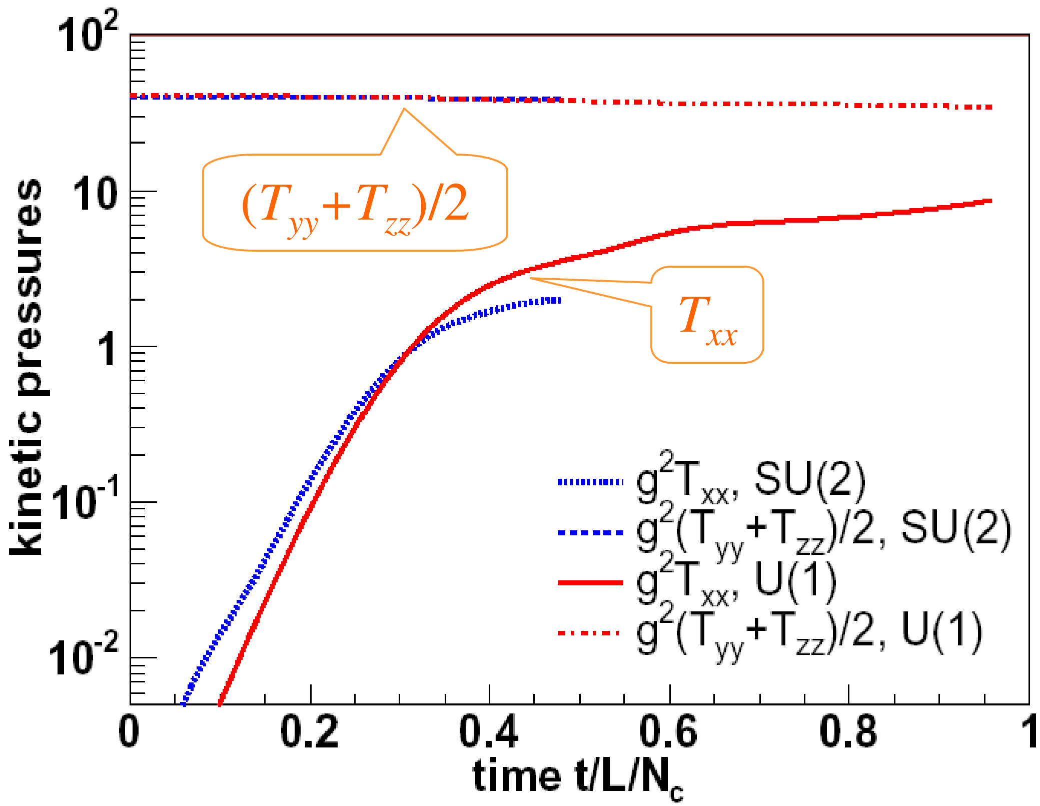
# Isotropization – numerical simulation

**Classical system of colored particles & fields**

$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

**Isotropy:**

$$T_{xx} = (T_{yy} + T_{zz})/2$$




A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

# Conclusion

**The scenario of instabilities driven equilibration provides a plausible solution to the fast equilibration problem of weakly coupled plasma**