Effective Coupling Constant of Plasmons

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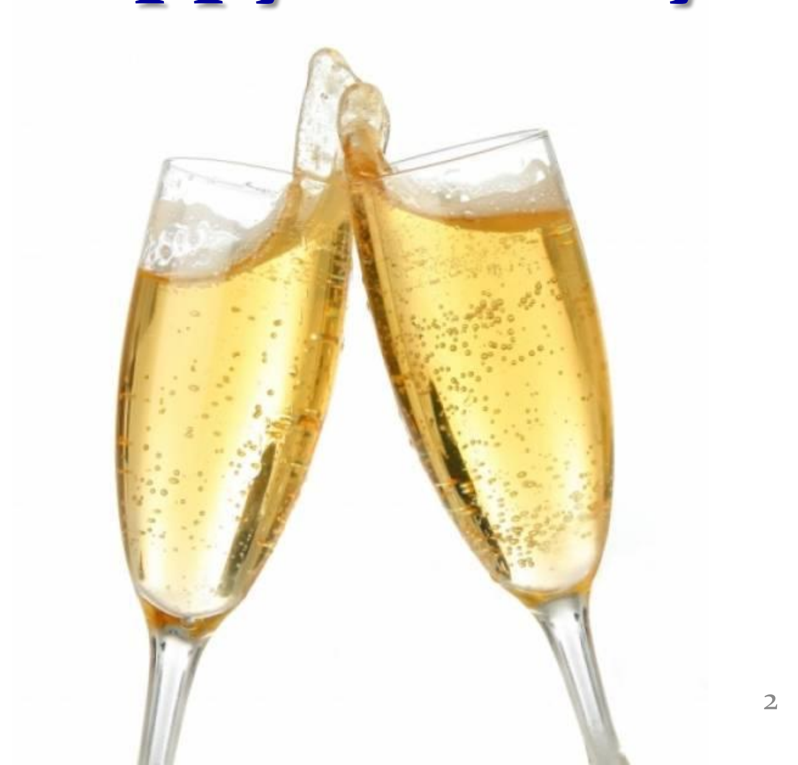
The lady’s man

Happy Birthday!

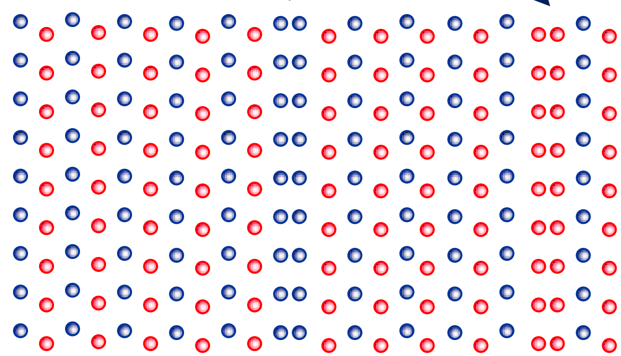


Background: plasmons

Longitudinal plasmons

charge fluctuation

$$\mathbf{E} \parallel \mathbf{k}$$



$$\mathbf{E}(t,\mathbf{r}) = \mathbf{E_0} \cos(\omega(\mathbf{k})\, t - \mathbf{k}\cdot\mathbf{r} + \varphi)$$

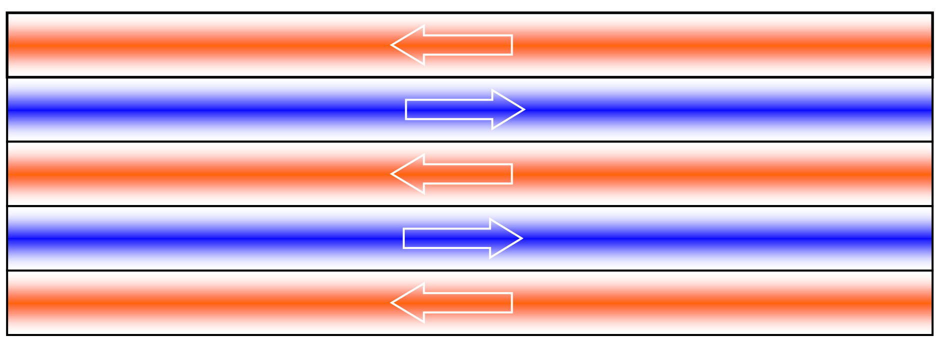
plasma or Langmuir frequency

$$\omega(\mathbf{k}) \underset{\mathbf{k}\to 0}{\approx} \omega_p \sim \boxed{eT}$$

ultrarelativsitic EM plasma

Transverse plasmons

current fluctuation



$$\mathbf{E} \perp \mathbf{k}$$

Background: plasma instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

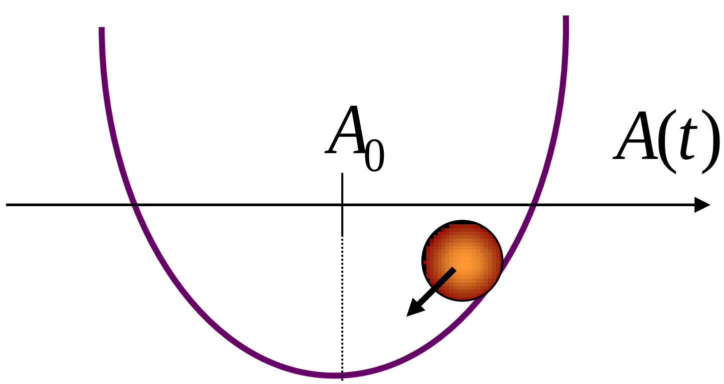
fluctuation

Instability

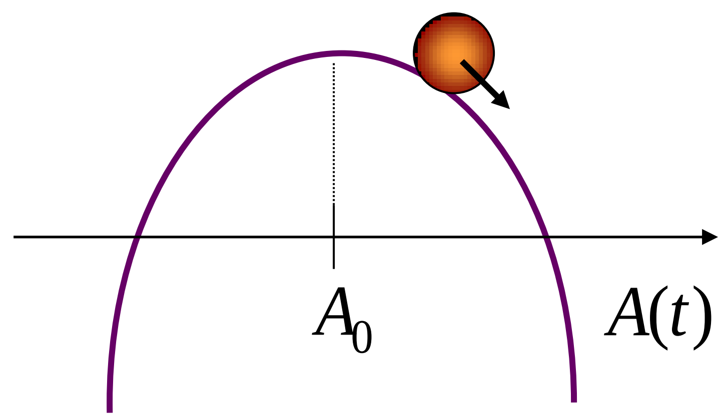
$$\delta A(t) \propto \mathrm{e}^{\gamma t}$$

$$\gamma > 0$$

stable configuration



unstable configuration



Background: plasma instabilities

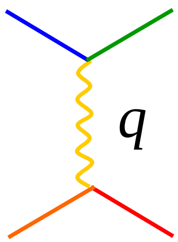
Anisotropic QGP is unstable

$$\delta A(t) \propto \mathrm{e}^{\gamma t}$$

$$\gamma \sim gT$$

T - hard momentum scale

Damping due to parton-parton scattering



hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Frequency of collisions

$$\nu_{\mathrm{hard}} \sim g^4 \ln(1/g)\, T$$

$$\nu_{\mathrm{soft}} \sim g^2 \ln(1/g)\, T$$

$$\text{If} \quad g^2 \ll 1 \quad \Rightarrow \quad \nu_{\mathrm{hard}} \ll \nu_{\mathrm{soft}} \ll \gamma$$

In weakly coupled plasma instabilities play an important role!

St. Mrówczyński, B. Schenke and M. Strickland, Phys. Rep. **682**, 1 (2017).

Background: running coupling constant

QED - Landau's pole

$$\alpha(q^2) \equiv \frac{e^2(q^2)}{4\pi} = \frac{3\pi}{\ln\left(\Lambda_{\mathrm{QED}}^2 / q^2\right)}$$

$$\Lambda_{\mathrm{QED}} \approx 10^{287}\,\mathrm{eV}$$

QCD - asymptotic freedom

$$\alpha(q^2) \equiv \frac{g^2(q^2)}{4\pi} = \frac{12\pi}{\left(33 - 2N_f\right)\ln\left(q^2 / \Lambda_{\mathrm{QCD}}^2\right)}$$

$$\Lambda_{\mathrm{QCD}} \approx 200\ \mathrm{MeV}$$

$$N_c = 3$$

What is $\alpha(q^2)$ for plasma collective modes?

What is q^2 ?

Plasmons in equilibrium QED plasma

Formulation of the problem

Plasmons – poles of propagator

Example of scalar fields

Dyson-Schwinger equation $$\Delta(k) = \Delta_0(k) + \Delta(k)\Pi(k)\Delta_0(k)$$

Free propagator $$\Delta_0(p) = \frac{1}{p^2 - m^2}$$

Resumed propagator $$\Delta(p) = \frac{1}{p^2 - m^2 - \Pi(p)}$$

Dispersion equation $$\Delta^{-1}(p) = p^2 - m^2 - \Pi(p) = 0$$

Photon propagator

Dyson-Schwinger equation

$$D(k) = D_0(k) + D(k)\Pi(k)D_0(k)$$

Free reatrded photon propagator

- General covariant gauge (GCG)

$$D_0^{\mu\nu}(k) = \frac{1}{k^2 + ik_0 0^+}\left(g^{\mu\nu} - (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right)$$

- Temporal axial ague (TAG)

$$D_0^{\mu\nu}(k) = \frac{1}{k^2 + ik_0 0^+}\left(g^{\mu\nu} + (1+\zeta)\frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + n^\mu k^\nu}{(k\cdot n)}\right)$$

$n^\mu = (1,0,0,0)$ - rest frame of the heat bath

Strict TAG $\zeta = 0$

$$\begin{cases} D_0^{00}(k) = D_0^{0i}(k) = D_0^{i0}(k) = 0 \\ D_0^{ij}(k) = -\dfrac{1}{k^2 + ik_0 0^+}\left(\delta^{ij} - \dfrac{k^i k^j}{k_0^2}\right) \end{cases}$$

Tensor decomposition for GCG

$n^{\mu} = (1,0,0,0)$ - rest frame of the heat bath $\qquad n_T^{\mu} \equiv \left(g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right) n_{\nu}$

Tensor basis

$$A^{\mu\nu}(k) \equiv g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} - \frac{n_T^{\mu} n_T^{\nu}}{n_T^2}, \qquad B^{\mu\nu}(k) \equiv \frac{n_T^{\mu} n_T^{\nu}}{n_T^2}$$

$$C^{\mu\nu}(k) \equiv k^{\mu} n_T^{\nu} + n_T^{\mu} k^{\nu}, \qquad E^{\mu\nu}(k) \equiv \frac{k^{\mu}k^{\nu}}{k^2}$$

$$D_0^{\mu\nu}(k) = \frac{1}{k^2 + ik_0 0^+} \left(A^{\mu\nu} + B^{\mu\nu} \right) + \frac{\zeta}{k^2 + ik_0 0^+} E^{\mu\nu}$$

$$k_{\mu} \Pi^{\mu\nu}(k) = 0 \quad \Rightarrow \quad \Pi^{\mu\nu}(k) = \Pi^{T}(k) A^{\mu\nu} + \Pi^{L}(k) B^{\mu\nu}$$

Dyson-Schwinger equation provides

$$D^{\mu\nu}(k) = D^{T}(k) A^{\mu\nu}(k) + D^{L}(k) B^{\mu\nu}(k) + \frac{\zeta}{k^2 + ik_0 0^+} E^{\mu\nu}(k)$$

$$D^{T,L}(k) \equiv \frac{1}{k^2 - \Pi^{T,L}(k)}$$

Tensor decomposition for TAG

$$T^{ij}(k) \equiv \delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}, \qquad L^{ij}(k) \equiv \frac{k^i k^j}{\mathbf{k}^2}$$

$$k_\mu \Pi^{\mu\nu}(k) = 0 \quad \Rightarrow \quad \Pi^{ij}(k) = \Pi^T(k) T^{ij} + \frac{k_0^2}{k^2} \Pi^L(k) L^{ij}$$

$$-D^{ij}(k) = D^T(k) T^{ij}(k) + \frac{k^2}{k_0^2} D^L(k) L^{ij}(k)$$

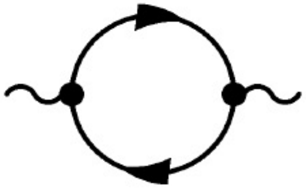
$$D^{T,L}(k) \equiv \frac{1}{k^2 - \Pi^{T,L}(k)}$$

Dispersion equation in GCG & TAG

$$k^2 - \Pi^{T,L}(k) = 0$$

Polarization tensor

Keldysh-Schwinger formalism



$$\Pi^{\mu\nu}(k) = 2e^2 \sum_{n=\pm 1} \int \frac{d^3 p}{(2\pi)^3} \frac{1 - n_f(|\mathbf{p}|)}{|\mathbf{p}|} \left. \frac{2 p^\mu p^\nu + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(k \cdot p)}{(p+k)^2 + i(p_0 + k_0) 0^+} \right|_{p_0 = n|\mathbf{p}|}$$

$$n_f(E) \equiv \frac{1}{e^{\beta E} + 1}$$

- $k_\mu \Pi^{\mu\nu}(k) = 0 \quad \Rightarrow \quad \Pi^{\mu\nu}(k) = \Pi^T(k) A^{\mu\nu}(k) + \Pi^L(k) B^{\mu\nu}(k)$

- $\Pi^{\mu\nu}(k) = \Pi^{\mu\nu}_{\text{vac}}(k) + \Pi^{\mu\nu}_{\text{med}}(k)$

UV divergent

UV & IR finite

Vacuum polarization tensor

One usually ignores the vacuum contribution as subleading when collective modes are studied but the vacuum makes the coupling run.

$$\Pi^{\mu\nu}_{\mathrm{vac}}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) P(k^2)$$

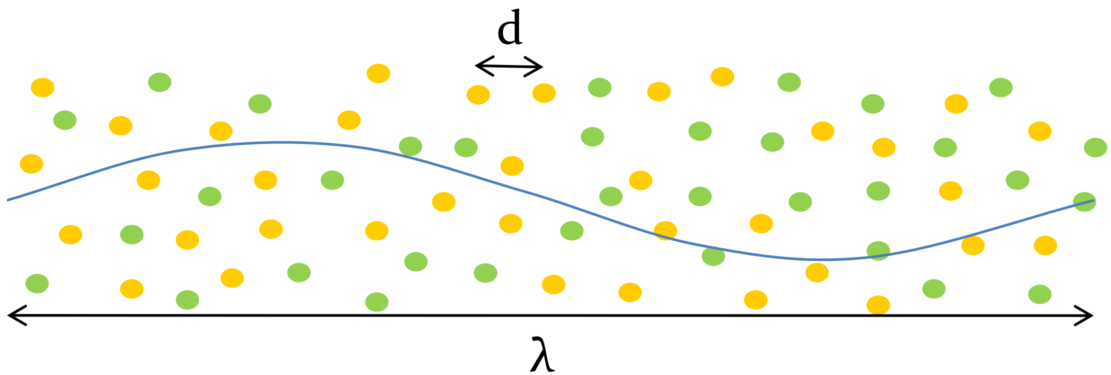
$$\Pi^{T}_{\mathrm{vac}}(k) = \Pi^{L}_{\mathrm{vac}}(k) = k^2 P(k^2)$$

$$P(k^2) = -\frac{e^2}{2\pi^2}\left[\frac{1}{6}\left(\frac{1}{\delta} - \gamma_E\right) - \int_0^1 dx\, x(1-x)\ln\left(-\frac{x(1-x)k^2}{4\pi M^2}\right)\right]$$

$m_e = 0$

Dimensionally regularized, divergent as $\delta \to 0$

Medium contribution



Expansion in $\left(\frac{k_0}{T}, \frac{|\mathbf{k}|}{T}\right)$

$$d << \lambda \quad \Leftrightarrow \quad (k_0, |\mathbf{k}|) << T$$

Leading order, HTL approximation

$$\begin{cases} \Pi^L_{\mathrm{HTL}}(k) = -\dfrac{k^2}{\mathbf{k}^2} m_D^2 \left[1 - \dfrac{k_0}{2|\mathbf{k}|}\left(\ln\left|\dfrac{|\mathbf{k}|+k_0}{|\mathbf{k}|-k_0}\right| - i\pi\,\Theta(-k^2)\right)\right] \\[2ex] \Pi^T_{\mathrm{HTL}}(k) = \dfrac{k_0^2}{2\mathbf{k}^2} m_D^2 \left[1 - \left(\dfrac{k_0}{2|\mathbf{k}|} - \dfrac{|\mathbf{k}|}{2k_0}\right)\left(\ln\left|\dfrac{|\mathbf{k}|+k_0}{|\mathbf{k}|-k_0}\right| - i\pi\,\Theta(-k^2)\right)\right] \end{cases}$$

$$m_D^2 \equiv \frac{e^2 T^2}{3}$$

V.P. Silin, Sov. Phys. JETP **11**, 1136 (1960) [Zh. Eksp. Teor. Fiz. **38**, 1577 (1960)].

Medium contribution cont.

Expansion in $\left(\dfrac{k_0}{T}, \dfrac{|\mathbf{k}|}{T}\right)$

Next-to-leading order

$$
\begin{cases}
\Pi^L_{\text{med}}(k) = \Pi^L_{\text{HTL}}(k) - \boxed{\dfrac{k^2}{12\pi^2}\ln\left(\dfrac{k^2}{T^2}\right)} + \ldots \\[2ex]
\Pi^T_{\text{med}}(k) = \Pi^T_{\text{HTL}}(k) - \boxed{\dfrac{k^2}{12\pi^2}\ln\left(\dfrac{k^2}{T^2}\right)} + \ldots
\end{cases}
$$

H.A. Weldon, Phys. Rev. D **26**, 1394 (1982).

Renormalization

$$\hat{D}^{T,L}(k,\mu) \equiv \frac{1}{Z_3(\mu)} D^{T,L}(k)$$

Renormalization condition

$$k^2 \to \mu^2 \quad \& \quad T=0 \qquad \Rightarrow \qquad \hat{D}^{T,L}(k,\mu) = \frac{1}{k^2}$$

The scale μ is arbitrary.

$$Z_3(\mu) = 1 + P(-\mu^2)$$

$$\hat{\Pi}^T_{\text{vac}}(k,\mu) = \hat{\Pi}^T_{\text{vac}}(k,\mu) = k^2 \hat{P}(k^2,\mu)$$

$$\hat{P}(k^2,\mu) \equiv P(k^2) - P(-\mu^2) = \frac{1}{12\pi^2} \ln\left(-\frac{k^2}{\mu^2}\right)$$

Charge renormalization

$$\hat{\alpha}(\mu) = Z_3(\mu)\,\alpha$$

$$\alpha \equiv \frac{e^2}{4\pi}$$

$$\mu \frac{d\hat{\alpha}(\mu)}{d\mu} = \beta(\mu)$$

At one-loop level:

$$\beta(\mu) = \frac{2}{3\pi}\hat{\alpha}(\mu)$$

$$\hat{\alpha}(\mu) = \frac{\hat{\alpha}(\mu_0)}{1 - \frac{\hat{\alpha}(\mu_0)}{3\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

Renormalized propagator

$$\hat{D}^{T,L}(k,\mu) = \frac{1}{k^2\left(1-\hat{P}(k^2,\mu)\right)-\Pi^{T,L}_{\text{med}}(k)}$$

$$\boxed{\hat{D}^{T,L}(k,\mu) = \frac{1}{k^2\left(1-\frac{\hat{\alpha}(\mu)}{3\pi}\ln\left(\frac{T^2}{\mu^2}\right)\right)-\hat{\alpha}(\mu)\,\pi^{T,L}(k)}}$$

$$\frac{k^2}{12\pi^2}\ln\left(\frac{|k^2|}{\mu^2}\right)-\frac{k^2}{12\pi^2}\ln\left(\frac{|k^2|}{T^2}\right)=\frac{k^2}{12\pi^2}\ln\left(\frac{T^2}{\mu^2}\right)$$

from vacuum

from medium

$\hat{\alpha}(\mu)\,\pi^{T,L}(k)$ includes all contributions to $\Pi^{T,L}(k,\mu)$ expect log terms

Scale dependent dispersion equation?

$$k^2\left(1-\frac{\hat{\alpha}(\mu)}{3\pi}\ln\left(\frac{T^2}{\mu^2}\right)\right)-\hat{\alpha}(\mu)\,\pi^{T,L}(k)=0$$

No! A physical quantity must be μ independent!

Plasmons – poles of

$$\hat{\alpha}(\mu)\,\hat{D}^{T,L}(k,\mu)$$

which is a renormlization-group invariant.

Coupling constant of plasmons

$$\hat{\alpha}(\mu)\hat{D}^{T,L}(k,\mu) = \frac{\hat{\alpha}(\mu)}{k^2\left(1 - \frac{\hat{\alpha}(\mu)}{3\pi}\ln\left(\frac{T^2}{\mu^2}\right)\right) - \hat{\alpha}(\mu)\,\pi^{T,L}(k)}$$

$$= \frac{\hat{\alpha}(T)}{k^2 - \hat{\alpha}(T)\,\pi^{T,L}(k)} = \hat{\alpha}(T)\hat{D}^{T,L}(k,T)$$

$$\hat{\alpha}(\mu) - \hat{\alpha}(\mu_0) = O\left(\hat{\alpha}^2(\mu)\right)$$

T is the scale of the effective coupling constant.

Conclusions

- Collective modes need to be defined through a renormalization-group invariant.

- Temperature is the scale of the effective coupling constant in equilibrium plasmas.

Outlook

- Anisotropic plasma

- QCD