Keldysh-Schwinger formalism & kinetic theory

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What is Keldysh-Schwinger formalism?

Formulation of quantum field theory (QFT) applicable to many-body (statistical) systems.

What for it is?

- To describe relativistic quantum statistical systems.
- To exploit machinery of QFT in description of nonrelativistic systems.

Outline

Lecture I – **Classical & quantum fields**

Lecture II – **Keldysh-Schwinger formalism**

Lecture III & IV – **From QFT to kinetic theory**

Lecture I

Classical & Quantum fields

- Lagrangian and Hamiltonian formalisms
- Canonical Quantization
- Path integral approach

Classical fields – Lagrange formalism

Lagrangian density of real or complex scalar fields

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2} m^2 \varphi^2 - \frac{g}{3!}\varphi^3$$

$$\mathcal{L} = (\partial^\mu \varphi^*)(\partial_\mu \varphi) - m^2 \varphi^* \varphi - \frac{g}{2!2!}(\varphi^* \varphi)^2$$

Principle of minimal action

$$\delta S = \delta \int d^4x\, \mathcal{L}(x) = 0$$

Euler-Lagrange equation

real field

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \qquad \Rightarrow \qquad [\partial^\mu \partial_\mu + m^2]\varphi = \frac{g}{2}\varphi^2$$

Klein-Gordon equation

Classical fields – Lagrange formalism

Euler-Lagrange equations

$$\begin{cases} \partial^{\mu} \dfrac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi)} - \dfrac{\partial \mathcal{L}}{\partial \varphi} = 0 \\ \partial^{\mu} \dfrac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi^*)} - \dfrac{\partial \mathcal{L}}{\partial \varphi^*} = 0 \end{cases}$$

complex field

$$\begin{cases} [\partial^{\mu}\partial_{\mu} + m^2]\varphi = \tfrac{1}{2} g \varphi^2 \varphi^* \\ [\partial^{\mu}\partial_{\mu} + m^2]\varphi^* = \tfrac{1}{2} g \varphi^{*2} \varphi \end{cases}$$

Classical fields – transformation properties

Why scalar field is called *scalar*?

Lorentz transformation

$$x \to x' \equiv \Lambda\, x$$

$$\varphi(x) \to \varphi'(x')$$

Postulate:

$$[\partial'^{\mu}\partial'_{\mu} + m^2]\varphi' = \tfrac{1}{2} g \varphi'^2$$

$\partial^{\mu}\partial_{\mu}, m^2$ scalars

$\Rightarrow$

Scalar field

$$\varphi'(x') = \varphi(\Lambda^{-1}x')$$

Classical fields – Noether theorem

Invariance of S under space-time translations $x^{\mu} \rightarrow x^{\mu} + \varepsilon^{\mu}$

$$\partial_{\mu} T^{\mu\nu} = 0$$ energy-momentum conservation

$$T^{\mu\nu} = (\partial^{\mu}\varphi)(\partial^{\nu}\varphi) - g^{\mu\nu}\mathcal{L}$$

$$T^{\mu\nu} = (\partial^{\mu}\varphi^{*})(\partial^{\nu}\varphi) + (\partial^{\mu}\varphi)(\partial^{\nu}\varphi^{*}) - g^{\mu\nu}\mathcal{L}$$

Invariance of S under $\varphi \rightarrow e^{i\theta}\varphi \approx \varphi + i\theta\varphi$

$$\partial_{\mu} j^{\mu} = 0$$ charge conservation

$$j^{\mu} = i\left(\varphi^{*}\partial^{\mu}\varphi - (\partial^{\mu}\varphi^{*})\varphi\right)$$

Classical fields – Noether theorem

Exercise - charge conservation

$$\delta\varphi = i\theta\varphi, \qquad \delta\varphi^* = -i\theta\varphi^*$$

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\delta(\partial^\mu\varphi) + \frac{\partial\mathcal{L}}{\partial\varphi^*}\delta\varphi^* + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}\delta(\partial^\mu\varphi^*) \right]$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial^\mu\delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi^*}\delta\varphi^* + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}\partial^\mu\delta\varphi^* \right]$$

$$= \int d^4x \left[\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)} \right)\delta\varphi + \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\delta\varphi \right) \right.$$

$$\left. + \left(\frac{\partial\mathcal{L}}{\partial\varphi^*} - \partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)} \right)\delta\varphi^* + \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}\delta\varphi^* \right) \right]$$

$$= \int d^4x\, \partial^\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}\delta\varphi^* \right) = \int d^4x\, \partial_\mu j^\mu_{\text{Noether}} = 0$$

$$j^\mu_{\text{Noether}} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^*)}\delta\varphi^* = i\theta\left((\partial^\mu\varphi^*)\varphi - \varphi^*\partial^\mu\varphi \right)$$

Classical fields – Hamiltonian formalism

Conjugate momentum

real field

$$\pi(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x)$$

complex field

$$\pi(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial \dot{\varphi}(x)} = \dot{\varphi}^*(x)$$

$$\pi^*(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial \dot{\varphi}^*(x)} = \dot{\varphi}(x)$$

Hamiltonian

$$H \equiv \int d^3x\, \mathcal{H}$$

$$\mathcal{H} = \begin{cases} \pi\dot{\varphi} - \mathcal{L} \\ \pi\dot{\varphi} + \pi^*\dot{\varphi}^* - \mathcal{L} \end{cases}$$

$$\mathcal{H} = \begin{cases} \pi^*\pi + \nabla\varphi^*\nabla\varphi + m^2\varphi^*\varphi + \frac{g}{2!2!}(\varphi^*\varphi)^2 \\ \frac{1}{2}\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2 + \frac{g}{3!}\varphi^3 \end{cases}$$

Classical fields – Hamiltonian formalism

Poisson bracket

$$\left\{A(t,\mathbf{x}), B(t,\mathbf{x}')\right\}_{\text{PB}} \equiv \int d^3x'' \left(\frac{\delta A(t,\mathbf{x})}{\delta\varphi(t,\mathbf{x}'')} \frac{\delta B(t,\mathbf{x}')}{\delta\pi(t,\mathbf{x}'')} - \frac{\delta A(t,\mathbf{x})}{\delta\pi(t,\mathbf{x}'')} \frac{\delta B(t,\mathbf{x}')}{\delta\pi(t,\mathbf{x}'')} \right)$$

Poisson bracket of canonical variables

$$\left\{\varphi(t,\mathbf{x}), \pi(t,\mathbf{x}')\right\}_{\text{PB}} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

Equations of motion

$$\begin{cases} \dot{\varphi}(x) = \left\{\varphi(x), H\right\}_{\text{PB}} = \dfrac{\delta H}{\delta\pi(x)} = \pi(x) \\ \dot{\pi}(x) = \left\{\pi(x), H\right\}_{\text{PB}} = -\dfrac{\delta H}{\delta\varphi(x)} = (\nabla^2 - m^2)\varphi(x) \end{cases}$$

Klein-Gordon equation

Canonical quantisation

Noninteracting fields

- $\varphi(x) \rightarrow \hat{\varphi}(x)$ field operators acting in Fock space
- $\{\ldots,\ldots\}_{\text{PB}} \rightarrow \frac{1}{i\hbar}[\ldots,\ldots]$
- construction of Fock space of states

Canonical quantisation

Commutation relations

$$\left[\hat{\varphi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right]=i\hbar\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

$$\left[\hat{\varphi}(t,\mathbf{x}),\hat{\varphi}(t,\mathbf{x}')\right]=\left[\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right]=0$$

real field

Conjugate momentum

$$\hat{\pi}(x)=\dot{\hat{\varphi}}(x)$$

Equation of motion

$$[\partial^{\mu}\partial_{\mu}+m^{2}]\hat{\varphi}(x)=0$$

Canonical quantisation

Solution of quation of motion

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left[e^{-ikx}\hat{a}(\mathbf{k}) + e^{ikx}\hat{a}^\dagger(\mathbf{k})\right]$$

$$\omega_{\mathbf{k}} \equiv \sqrt{m^2 + \mathbf{k}^2}$$

$$\begin{cases} \left[\hat{\varphi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}')\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \\ \left[\hat{\varphi}(t,\mathbf{x}), \hat{\varphi}(t,\mathbf{x}')\right] = 0 \\ \left[\hat{\pi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}')\right] = 0 \end{cases} \Leftrightarrow \begin{cases} \left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\ \left[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')\right] = 0 \\ \left[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = 0 \end{cases}$$

Hamiltonian

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_{\mathbf{k}}}{2}\left[\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) + \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\right]$$

Canonical quantisation

Discretisation

field is periodic

$$\hat{\varphi}(t,\mathbf{x}) = \hat{\varphi}(t,\mathbf{x}+\mathbf{e}_k L) \qquad \begin{cases} \mathbf{e}_1 = (1,0,0) \\ \mathbf{e}_2 = (0,1,0) \\ \mathbf{e}_3 = (0,0,1) \end{cases}$$

$$\int \frac{d^3k}{(2\pi)^3}\cdots \to \frac{1}{L^3}\sum_i \cdots \qquad (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \to L^3\delta^{ij}$$

$$\hat{a}_i \equiv \frac{1}{\sqrt{L^3}}\hat{a}(\mathbf{k}_i) \qquad \hat{a}_i^\dagger \equiv \frac{1}{\sqrt{L^3}}\hat{a}^\dagger(\mathbf{k}_i)$$

Commutation relations

$$\begin{cases} \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta^{ij} \\ \left[\hat{a}_i, \hat{a}_j\right] = 0 \\ \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0 \end{cases}$$

Hamiltonian

$$\hat{H} = \sum_i \frac{\omega_i}{2}\left[\hat{a}_i\hat{a}_i^\dagger + \hat{a}_i^\dagger\hat{a}_i\right] = \sum_i \omega_i\left[\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right]$$

Construction of Fock space

Postulate $\exists |E\rangle \qquad \hat{H}|E\rangle = E|E\rangle$

$$\hat{H}\hat{a}_i|E\rangle = (E-\omega_i)\hat{a}_i|E\rangle \quad \Rightarrow \quad \hat{a}_i|E\rangle = |E-\omega_i\rangle$$

$$\hat{H}\hat{a}_i^\dagger|E\rangle = (E+\omega_i)\hat{a}_i^\dagger|E\rangle \quad \Rightarrow \quad \hat{a}_i^\dagger|E\rangle = |E+\omega_i\rangle$$

$$\left[\hat{H},\hat{a}_i\right] = -\omega_i\hat{a}_i \qquad \left[\hat{H},\hat{a}_i^\dagger\right] = \omega_i\hat{a}_i^\dagger$$

Positive definitness of *H*

$$\forall|\alpha\rangle \qquad \langle\alpha|\hat{H}|\alpha\rangle \geq 0$$

Existence of vacuum (ground state)

$$\hat{a}_i\,|0\rangle = 0 \qquad \langle 0|\hat{a}_i^\dagger = 0$$

Construction of Fock space

$$\hat{H}|0\rangle = \sum_i \omega_i \left[\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right] |0\rangle = \sum_i \frac{\omega_i}{2} |0\rangle = \infty$$

Normal ordering

$$\hat{H} = \sum_i \frac{\omega_i}{2} \left[\hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i \right] \quad \rightarrow \quad \hat{H} = \sum_i \omega_i \hat{a}_i^\dagger \hat{a}_i \qquad \langle 0 | \hat{H} | 0 \rangle = 0$$

$$\hat{a}_i |n_i\rangle = C_{n_i} |n_i - 1\rangle \qquad \hat{H}|n_i\rangle = n_i \omega_i |n_i\rangle \qquad \langle n_i | n_j \rangle = \delta^{ij}$$

$$C_{n_i} = ?$$

$$\langle n_i | \hat{a}_i^\dagger \hat{a}_i | n_i \rangle = \begin{cases} |C_{n_i}|^2 \langle n_i - 1 | n_i - 1 \rangle = |C_{n_i}|^2 \\ \langle n_i | \dfrac{\sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j}{\omega_i} | n_i \rangle = \langle n_i | \dfrac{\hat{H}}{\omega_i} | n_i \rangle = n_i \langle n_i | n_i \rangle = n_i \end{cases}$$

$$|C_{n_i}|^2 = n_i \quad \Rightarrow \quad C_{n_i} = \sqrt{n_i}, \quad C_{n_i} \in R$$

Construction of Fock space

$$\hat{a}_i^\dagger |n_i\rangle = D_{n_i} |n_i + 1\rangle \qquad \hat{H} |n_i\rangle = n_i \omega_i |n_i\rangle \qquad \langle n_i | n_j\rangle = \delta^{ij}$$

$$D_{n_i} = ?$$

$$\langle n_i | \hat{a}_i \hat{a}_i^\dagger | n_i\rangle = \begin{cases} |D_{n_i}|^2 \langle n_i + 1 | n_i + 1\rangle = |D_{n_i}|^2 \\ \langle n_i | \hat{a}_i^\dagger \hat{a}_i + 1 | n_i\rangle = \langle n_i | \hat{a}_i^\dagger \hat{a}_i | n_i\rangle + \langle n_i | n_i\rangle = n_i + 1 \end{cases}$$

$$|D_{n_i}|^2 = n_i + 1 \quad \Rightarrow \quad D_{n_i} = \sqrt{n_i + 1}$$

$$|n_i\rangle = \frac{1}{\sqrt{n_i\,!}} (\hat{a}_i^\dagger)^{n_i} |0\rangle$$

Construction of Fock space

$$\hat{a}_i \left| n_1, n_2, \ldots n_i, \ldots \right\rangle = \sqrt{n_i} \left| n_1, n_2, \ldots n_i - 1, \ldots \right\rangle$$

$$\hat{a}_i^\dagger \left| n_1, n_2, \ldots n_i, \ldots \right\rangle = \sqrt{n_i + 1} \left| n_1, n_2, \ldots n_i + 1, \ldots \right\rangle$$

$$\left| n_1, n_2, n_3, \ldots \right\rangle = \frac{1}{\sqrt{n_1! n_2! n_3! \ldots}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} (\hat{a}_3^\dagger)^{n_3} \ldots \left| 0 \right\rangle$$

$$\hat{H} \left| n_1, n_2, n_3, \ldots \right\rangle = \sum_i \omega_i n_i \left| n_1, n_2, n_3, \ldots \right\rangle$$

Time evolution & perturbative expansion

Temporal evolution

$$\left|\psi(t_f)\right\rangle = T\exp\left(-i\int_{t_i}^{t_f} dt\,\hat{H}(t)\right)\left|\psi(t_i)\right\rangle$$

Perturbative expansion

$$e^{\hat{A}}e^{\hat{B}} \neq e^{\hat{A}+\hat{B}}$$

$$T\exp\left(-i\int_{t_i}^{t_f} dt\,\hat{H}(t)\right) = 1 - i\int_{t_i}^{t_f} dt\,\hat{H}(t) + T\int_{t_i}^{t_f} dt\int_{t_i}^{t_f} dt'\,\hat{H}(t)\hat{H}(t') + \ldots$$

Interaction picture

$$\hat{H}_{\text{int}}(t) = \begin{cases} \dfrac{g}{3!}\displaystyle\int d^3x\,\varphi^3(t,\mathbf{x}) \\ \dfrac{g}{2!2!}\displaystyle\int d^3x\left(\varphi^*(t,\mathbf{x})\,\varphi(t,\mathbf{x})\right)^2 \end{cases}$$

Transition matrix elements

$$t_i \to -\infty, \qquad t_f \to \infty$$

$$S_{if} = \langle f \,|\, T \exp\left(-i \int_{-\infty}^{\infty} dt\, \hat{H}(t) \right) |\, i \rangle$$

$$|\, n_1, n_2, n_3, \ldots \rangle = \frac{1}{\sqrt{n_1 ! n_2 ! n_3 ! \ldots}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} (\hat{a}_3^\dagger)^{n_3} \ldots |\, 0 \rangle$$

Wick theorem

For odd n

$$\langle 0|T\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)|0\rangle = 0$$

For even n

$$\langle 0|T\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)|0\rangle =$$

$$=\langle 0|T\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\right)|0\rangle\langle 0|T\left(\hat{\varphi}(x_3)\hat{\varphi}(x_4)\right)|0\rangle\ldots\langle 0|T\left(\hat{\varphi}(x_{n-1})\hat{\varphi}(x_n)\right)|0\rangle$$

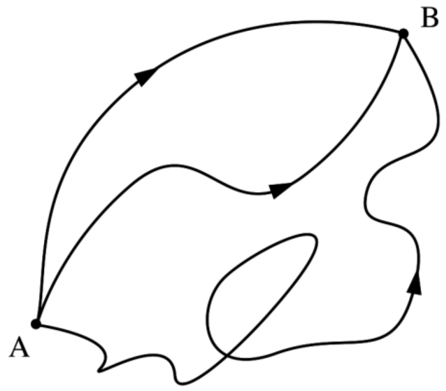
$$+\text{ permutations}$$

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left[e^{-ikx}\hat{a}(\mathbf{k}) + e^{ikx}\hat{a}^\dagger(\mathbf{k})\right]$$

$$\hat{a}(\mathbf{k})|0\rangle = 0 = \langle 0|\hat{a}_i^\dagger(\mathbf{k})$$

$$\begin{cases} \left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\ \left[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')\right] = 0 \\ \left[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = 0 \end{cases}$$

Path integral methods



Quantum mechanics

$$\langle x_B, t_B | x_A, t_A \rangle = N \int \mathcal{D}x(t)\, e^{iS}$$

$$S \equiv \int_{t_A}^{t_B} dt\, L(t), \qquad x(t_A) = x_A, \quad x(t_B) = x_B$$

QFT: vacuum-vacuum transition amplitude

$$\langle 0 | T \exp\left(-i \int_{-\infty}^{\infty} dt\, \hat{H}(t) \right) | 0 \rangle = N \int \mathcal{D}\varphi\, e^{i \int d^4x \mathcal{L}(x)}$$

$$\varphi(t \to \pm\infty, \mathbf{x}) = 0$$

Generating functional

$$W[J] = N\int \mathcal{D}\varphi \, e^{i\int d^4x \mathcal{L}(x)} \qquad \varphi(t \to \pm\infty, \mathbf{x}) = 0$$

$$\mathcal{L}(x) = \underbrace{\tfrac{1}{2}\partial^\mu \varphi(x)\partial_\mu \varphi(x) - \tfrac{1}{2} m^2 \varphi^2(x)}_{\mathcal{L}_0(x)} + \mathcal{L}_{\text{int}}(x) + J(x)\varphi(x)$$

$$N^{-1} = W[J=0]$$

$$i\Delta(x_1, x_2, \ldots x_n) \equiv N\int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n) e^{i\int d^4x \mathcal{L}(x)}\Big|_{J=0}$$

$$i\Delta(x_1, x_2, \ldots x_n) = (-i)^n \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} W[J]\Big|_{J=0}$$

Perturbative expansion

$$W[J] = N\int \mathcal{D}\varphi \exp\left(iS_0[\varphi] + iS_{\text{int}}[\varphi] + i\int d^4x\, \varphi j\right)$$

$$S_{\text{int}}[\varphi] \equiv \int d^4x\, \mathcal{L}_{\text{int}}(x) = \frac{g}{3!}\int d^4x\, \varphi^3(x)$$

$$\exp\left(ax^2 + bx^3 + jx\right) = \exp\left(b\frac{d^3}{dj^3}\right)\exp\left(ax^2 + jx\right)$$

$$\exp\left(b\frac{d^3}{dj^3}\right)\exp(jx) = \left(1 + b\frac{d^3}{dj^3} + \frac{1}{2!}\left(b\frac{d^3}{dj^3}\right)^2 + \ldots\right)\exp(jx)$$

$$= \left(1 + bx^3 + \frac{1}{2!}\left(bx^3\right)^2 + \ldots\right)\exp(jx) = \exp\left(bx^3\right)\exp(jx)$$

$$= \exp\left(bx^3 + jx\right)$$

$$W[J] = N\exp\left(iS_{\text{int}}\left[\frac{\delta}{i\delta J}\right]\right)\int \mathcal{D}\varphi \exp\left(iS_0[\varphi] + i\int d^4x\, \varphi j\right)$$

Perturbative expansion

$$W[J] = N \exp\left(iS_{\text{int}}\left[\frac{\delta}{i\delta J} \right] \right) W_0[J]$$

$$W_0[J] = \int \mathcal{D}\varphi \exp\left(iS_0[\varphi] + i\int d^4x\, \varphi\, j \right)$$

Explicit form of free generating functional

$$W_0[J] = \exp\left(\tfrac{i}{2} \int d^4x \int d^4y\, j(x) \Delta^F(x-y) j(y) \right)$$

$$\exp\left(iS_{\text{int}}\left[\frac{\delta}{i\delta J} \right] \right) = 1 + iS_{\text{int}}\left[\frac{\delta}{i\delta J} \right] + \frac{1}{2!}\left(iS_{\text{int}}\left[\frac{\delta}{i\delta J} \right] \right)^2 + \ldots$$

$$S_{\text{int}}\left[\frac{\delta}{i\delta J} \right] = \frac{g}{3!} \int d^4x \frac{\delta^3}{\delta j^3(x)}$$

Perturbative expansion

Exercise

$$i\Delta(x_1,x_2)=(-i)^n\frac{\delta^n}{\delta J(x_1)\delta J(x_2)}W[J]\Big|_{J=0}$$

$$W[J]=N\exp\left(iS_{\text{int}}\left[\frac{\delta}{i\delta J}\right]\right)W_0[J]=$$

$$N\left[1+\frac{g}{3!}\int d^4x_3\frac{\delta^3}{\delta j^3(x_3)}+\frac{1}{2!}\left(\frac{g}{3!}\int d^4x_4\frac{\delta^3}{\delta j^3(x_4)}\right)\left(\frac{g}{3!}\int d^4x_5\frac{\delta^3}{\delta j^3(x_3)}\right)+\ldots\right]$$

$$\exp\left(\tfrac{i}{2}\int d^4x\int d^4y\; j(x)\Delta^F(x-y)\,j(y)\right)$$

Lecture II

Keldysh-Schwinger formalism

- Contour & real-time Green's functions
- Wigner transformation
- Free Green's functions
- Perturbative expansion

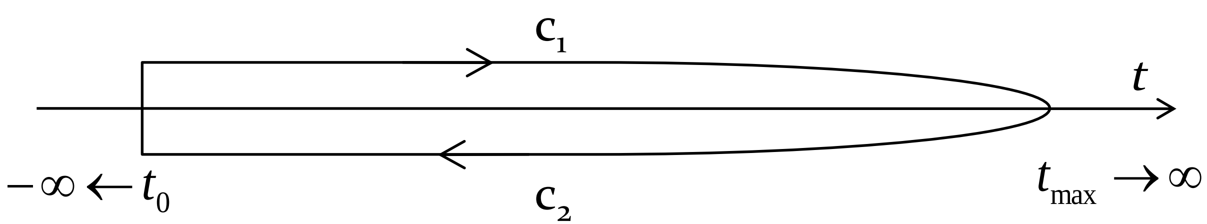
Contour Green's function

$$i\Delta(x,y) \equiv \left\langle \tilde{T}\varphi(x)\varphi^{\dagger}(y) \right\rangle$$

$$\left\langle \ldots \right\rangle \equiv \frac{\mathrm{Tr}[\hat{\rho}(t_0)\ldots]}{\mathrm{Tr}[\hat{\rho}(t_0)]}$$

Ordering along the contour

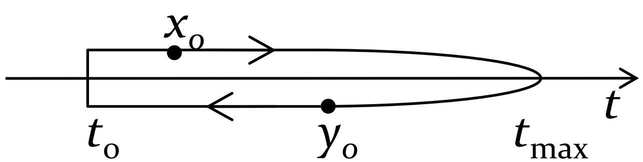
$$\tilde{T}A(x)B(y) = \theta(x_0, y_0)A(x)B(y) \pm \theta(y_0, x_0)B(y)A(x)$$



Green’s functions of real time arguments

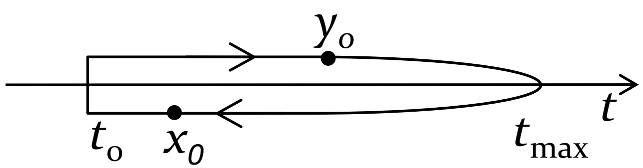
no ordering

$$i\Delta^{>}(x,y)=\left\langle \varphi(x)\varphi^{\dagger}(y)\right\rangle$$



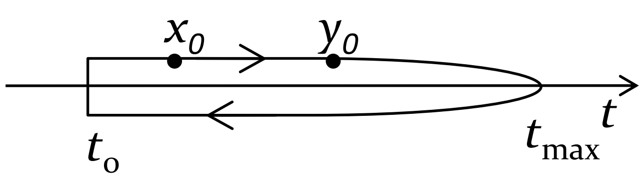
no ordering

$$i\Delta^{<}(x,y)=\left\langle \varphi^{\dagger}(x)\varphi(y)\right\rangle$$



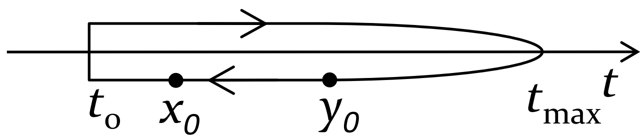
chronological ordering

$$i\Delta^{c}(x,y)=\left\langle T^{c}\varphi(x)\varphi^{\dagger}(y)\right\rangle$$



antichronological ordering

$$i\Delta^{a}(x,y)=\left\langle T^{a}\varphi(x)\varphi^{\dagger}(y)\right\rangle$$



Physical meaning of Green's functions

$$i\Delta^{>}(x,y) = \left\langle \varphi(x)\varphi^{\dagger}(y) \right\rangle$$

phase-space denisties

$$i\Delta^{<}(x,y) = \left\langle \varphi^{\dagger}(x)\varphi(y) \right\rangle$$

Wigner transformation

$$\Delta^{>}(X,p) = \int d^4u\, e^{ipu} \Delta^{>}(X + \tfrac{1}{2}u, X - \tfrac{1}{2}u)$$

$$X \equiv \tfrac{1}{2}(x+y), \qquad u \equiv x - y$$

$$\Delta^{>}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \Delta^{>}(\tfrac{1}{2}(x+y), p)$$

Physical meaning of Green's functions

Exercise

$$\Delta^{>}(x,y)=\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\Delta^{>}(\tfrac{1}{2}(x+y),p)$$

$$\begin{aligned}
j^{\mu}(x) &= i\left\langle \varphi(x)\partial^{\mu}\varphi^{\dagger}(x)-(\partial^{\mu}\varphi(x))\varphi^{\dagger}(x)\right\rangle \\
&= \lim_{y\to x}\left[-\partial_x^{\mu}\Delta^{>}(x,y)+\partial_y^{\mu}\Delta^{>}(x,y)\right] \\
&= \lim_{y\to x}\left[\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\left(ip^{\mu}-\tfrac{1}{2}\partial_X^{\mu}\right)\Delta^{>}(X,p)-\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\left(-ip^{\mu}-\tfrac{1}{2}\partial_X^{\mu}\right)\Delta^{>}(X,p)\right] \\
&= \lim_{y\to x}\left[2\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}p^{\mu}i\Delta^{>}(X,p)\right]
\end{aligned}$$

$$X\equiv\tfrac{1}{2}(x+y)$$

$$j^{\mu}(X)=2\int\frac{d^4p}{(2\pi)^4}p^{\mu}i\Delta^{>}(X,p)$$

Physical meaning of Green's functions

$$\begin{cases} j^\mu(X) = 2\int \frac{d^4p}{(2\pi)^4}\, p^\mu i\Delta^>(X,p) \\ T_0^{\mu\nu}(X) = 2\int \frac{d^4p}{(2\pi)^4}\, p^\mu p^\nu i\Delta^>(X,p) \end{cases}$$

$$p^2 \neq m^2$$

$i\Delta^>(X,p)$ not positive definite

in kinetic theory

$$\begin{cases} j^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E_{\mathbf{p}}}\left[f(X,\mathbf{p}) - \bar{f}(X,\mathbf{p})\right] \\ T_0^{\mu\nu}(X) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_{\mathbf{p}}}\left[f(X,\mathbf{p}) + \bar{f}(X,\mathbf{p})\right] \end{cases}$$

$$p^\mu = (E_{\mathbf{p}}, \mathbf{p})$$

$$E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$$

$$p^2 = m^2$$

$$f(X,\mathbf{p}),\ \bar{f}(X,\mathbf{p}) \geq 0$$

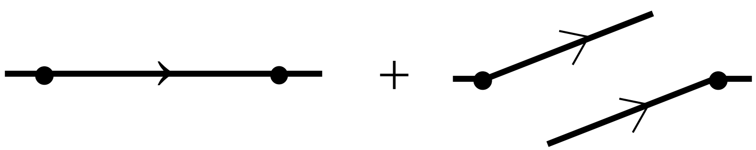
Physical meaning of Green's functions

$$i\Delta^{c}(x,y)=\left\langle T^{c}\varphi(x)\varphi^{\dagger}(y)\right\rangle$$ Feynman propagator

$$i\Delta^{a}(x,y)=\left\langle T^{a}\varphi(x)\varphi^{\dagger}(y)\right\rangle$$ antiFeynman propagator

in vacuum

in medium



$p^2 \neq m^2$ $\qquad$ $p^2 = m^2$

Free Green's functions of thermal system

Equilibrium system

$$\hat{\rho} = e^{-\beta \hat{H}}$$

Countour Green's function

$$i\Delta(x,y) \equiv \frac{\mathrm{Tr}[e^{-\beta \hat{H}} \tilde{T} \varphi(x) \varphi^{\dagger}(y)]}{\mathrm{Tr}[e^{-\beta \hat{H}}]}$$

Partition function

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$$

Discretized system

$$\hat{\varphi}(x) = \frac{1}{L^3} \sum_i \frac{1}{\sqrt{2\omega_i}} \left[e^{-ik_i x} \hat{a}_i + e^{ik\, x} \hat{a}_i^{\dagger} \right]$$

$$\begin{cases} \left[\hat{a}_i, \hat{a}_j^{\dagger} \right] = \delta^{ij} \\ \left[\hat{a}_i, \hat{a}_j \right] = 0 \\ \left[\hat{a}_i^{\dagger}, \hat{a}_j^{\dagger} \right] = 0 \end{cases}$$

$$\hat{H} = \sum_i \frac{\omega_i}{2} \left[\hat{a}_i \hat{a}_i^{\dagger} + \hat{a}_i^{\dagger} \hat{a}_i \right] \to \sum_i \omega_i \hat{a}_i^{\dagger} \hat{a}_i$$

Partition function

Exercise

$$Z = \mathrm{Tr}[e^{-\beta\hat{H}}] = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1, n_2, \ldots | e^{-\beta\hat{H}} | n_1, n_2, \ldots\rangle$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1, n_2, \ldots | \exp\left(-\beta\sum_i n_i\omega_i\right) | n_1, n_2, \ldots\rangle$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\exp\left(-\beta\sum_i n_i\omega_i\right)$$

$$= \sum_{n_1=0}^{\infty} e^{-\beta n_1\omega_1}\sum_{n_2=0}^{\infty} e^{-\beta n_2\omega_2}\ldots = \frac{1}{1-e^{-\beta\omega_1}}\frac{1}{1-e^{-\beta\omega_2}}\ldots$$

$$Z = \exp\left(-\sum_i \ln\left(1-e^{-\beta\omega_i}\right)\right) \quad \Rightarrow \quad Z = \exp\left(-V\int\frac{d^3k}{(2\pi)^3}\ln\left(1-e^{-\beta\omega_{\mathbf{k}}}\right)\right)$$

$$V = L^3 \qquad \omega_{\mathbf{k}} \equiv \sqrt{m^2+\mathbf{k}^2}$$

Free Green's functions of thermal system

$$\langle n_1, n_2, \ldots, n_i, \ldots | \hat{a}_i \hat{a}_j^\dagger | n_1, n_2, \ldots, n_j, \ldots \rangle = \sqrt{(n_i+1)(n_j+1)}\ \delta^{ij}$$

$$\langle n_1, n_2, \ldots, n_i, \ldots | \hat{a}_i^\dagger \hat{a}_j | n_1, n_2, \ldots, n_j, \ldots \rangle = \sqrt{n_i n_j}\ \delta^{ij}$$

System is translationally invariant $\quad \Delta^{>}(x,y) = \Delta^{>}(x-y)$

Wigner transformation becomes Fourier transformation

$$\begin{cases} \Delta^{>}(k) = \int d^4(x-y)\, e^{ik(x-y)} \Delta^{>}(x-y) \\ \Delta^{>}(x-y) = \int \dfrac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta^{>}(k) \end{cases}$$

Free Green's functions of thermal system

Phase-space denisties

$$\begin{cases} i\Delta^{>}(k) = \dfrac{\pi}{\omega_{\mathbf{k}}}\left[\delta(\omega_{\mathbf{k}} - k_0)\left(f(\omega_{\mathbf{k}}) + 1\right) + \delta(\omega_{\mathbf{k}} + k_0) f(\omega_{\mathbf{k}})\right] \\ i\Delta^{<}(k) = \dfrac{\pi}{\omega_{\mathbf{k}}}\left[\delta(\omega_{\mathbf{k}} - k_0) f(\omega_{\mathbf{k}}) + \delta(\omega_{\mathbf{k}} + k_0)\left(f(\omega_{\mathbf{k}}) + 1\right)\right] \end{cases}$$

$$f(\omega_{\mathbf{k}}) \equiv \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1} \qquad \omega_{\mathbf{k}} \equiv \sqrt{m^2 + \mathbf{k}^2}$$

- $i\Delta^{>}(k), i\Delta^{<}(k) > 0$ positive devinite
- $k^2 \neq m^2 \quad \Rightarrow \quad \Delta^{>}(k) = \Delta^{<}(k) = 0$ on mass-shell

Free Green's functions of thermal system

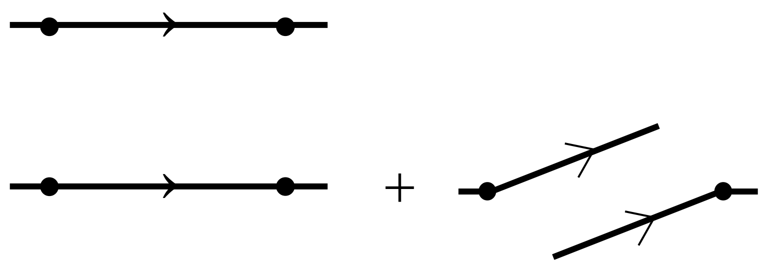
Propagaors

$$
\begin{cases}
i\Delta^{c}(k) = \dfrac{i}{k^2 - m^2 + i0^+} + \dfrac{\pi}{\omega_{\mathbf{k}}} f(\omega_{\mathbf{k}})\left[\delta(\omega_{\mathbf{k}} - k_0) + \delta(\omega_{\mathbf{k}} + k_0)\right] \\[2em]
i\Delta^{a}(k) = \dfrac{-i}{k^2 - m^2 - i0^+} + \dfrac{\pi}{\omega_{\mathbf{k}}} f(\omega_{\mathbf{k}})\left[\delta(\omega_{\mathbf{k}} - k_0) + \delta(\omega_{\mathbf{k}} + k_0)\right]
\end{cases}
$$

$$
f(\omega_{\mathbf{k}}) \equiv \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1}
$$

in vacuum

in medium



$k^2 \neq m^2$ $\qquad$ $k^2 = m^2$

Non-equilibrium free Green's functions

Equations of motion

$$\begin{cases} [\Box + m^2]\varphi(x) = 0 \\ [\Box + m^2]\varphi^\dagger(x) = 0 \end{cases} \Rightarrow \begin{cases} [\Box_x + m^2]\Delta^{>(<)}(x,y) = 0 \\ [\Box_y + m^2]\Delta^{>(<)}(x,y) = 0 \\ [\Box_x + m^2]\Delta^{c(a)}(x,y) = \mp\delta^{(4)}(x-y) \\ [\Box_y + m^2]\Delta^{c(a)}(x,y) = \mp\delta^{(4)}(x-y) \end{cases}$$

$$\Box \equiv \partial^\mu \partial_\mu$$

$$\frac{d}{dt}\theta(t) = \delta(t)$$

$$\varphi(t,\mathbf{x})\dot{\varphi}(t,\mathbf{y}) - \dot{\varphi}(t,\mathbf{y})\varphi(t,\mathbf{x}) = \varphi(t,\mathbf{x})\pi(t,\mathbf{y}) - \pi(t,\mathbf{y})\varphi(t,\mathbf{x}) = \left[\varphi(t,\mathbf{x}), \pi(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

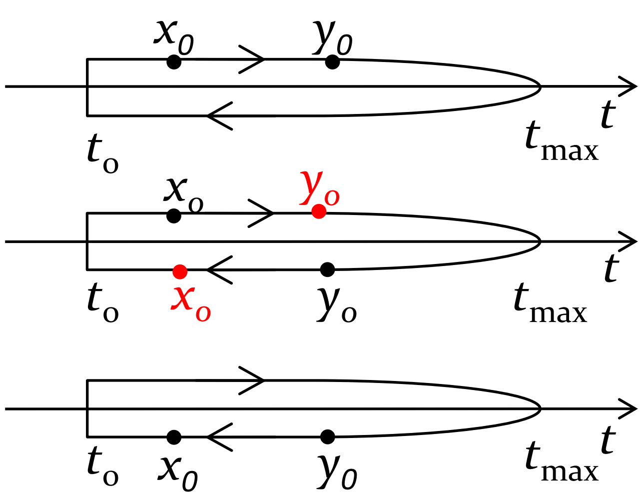
Non-equilibrium free Green's functions

Contour Green's function

$$\begin{cases} [\Box_x + m^2]\Delta(x,y) = -\delta_C^{(4)}(x,y) \\ [\Box_y + m^2]\Delta(x,y) = -\delta_C^{(4)}(x,y) \end{cases}$$

Contour delta function

$$\delta_C^{(4)}(x,y) = \begin{cases} \delta^{(4)}(x-y) \\ 0 \\ -\delta^{(4)}(x-y) \end{cases}$$



Non-equilibrium free Green's functions

Equations of motion of phase-space densities

$$\Delta(X,p)=\int d^4u\, e^{ipu}\Delta(X+\tfrac{1}{2}u, X-\tfrac{1}{2}u) \qquad X\equiv\tfrac{1}{2}(x+y), \qquad u\equiv x-y$$

$$\begin{cases}[\tfrac{1}{4}\partial^2 - ip_\mu\partial^\mu - p^2+m^2]\Delta^{>(<)}(X,p)=0 \\ [\tfrac{1}{4}\partial^2 + ip_\mu\partial^\mu - p^2+m^2]\Delta^{>(<)}(X,p)=0\end{cases}$$

Transport equation

$$p_\mu\partial^\mu\Delta^{>(<)}(X,p)=0$$

Mass-shell equation

$$[\tfrac{1}{4}\partial^2 - p^2+m^2]\Delta^{>(<)}(X,p)=0$$

Quasi-particle approximation

$$\frac{1}{m^2}\left|\partial^2\Delta^{>(<)}(X,p)\right| << \left|\Delta^{>(<)}(X,p)\right| \qquad \Rightarrow \qquad [p^2-m^2]\Delta^{>(<)}(X,p)=0$$

$$\Delta^{>(<)}(X,p)\sim\delta(p^2-m^2)$$

Non-equilibrium free Green's functions

Equations of motion of propagators

$$\begin{cases} [\frac{1}{4}\partial^2 - ip_\mu\partial^\mu - p^2 + m^2]\Delta^{c(a)}(X,p) = \mp 1 \\ [\frac{1}{4}\partial^2 + ip_\mu\partial^\mu - p^2 + m^2]\Delta^{c(a)}(X,p) = \mp 1 \end{cases}$$

Transport equation

$$p_\mu\partial^\mu\Delta^{c(a)}(X,p) = 0$$

Mass-shell equation

$$[\frac{1}{4}\partial^2 - p^2 + m^2]\Delta^{c(a)}(X,p) = \mp 1$$

Quasi-particle approximation

$$\frac{1}{m^2}\left|\partial^2\Delta^{c(a)}(X,p)\right| << \left|\Delta^{c(a)}(X,p)\right| \quad \Rightarrow \quad [p^2 - m^2]\Delta^{c(a)}(X,p) = \mp 1$$

$$\Delta^{c(a)}(X,p) = \mp\frac{1}{p^2 - m^2} + \delta(p^2 - m^2)f(X,p)$$

Perturbative expansion – operator approach

Wick theorem

For odd n

$$\left\langle \tilde{T}\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)\right\rangle = 0$$

For even n

$$\left\langle \tilde{T}\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)\right\rangle =$$

$$= \left\langle \tilde{T}\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\right)\right\rangle \left\langle \tilde{T}\left(\hat{\varphi}(x_3)\hat{\varphi}(x_4)\right)\right\rangle \ldots \left\langle \tilde{T}\left(\hat{\varphi}(x_{n-1})\hat{\varphi}(x_n)\right)\right\rangle$$

$$+ \text{permutations}$$

$$\langle \ldots \rangle \equiv \frac{\mathrm{Tr}[\hat{\rho}(t_0)\ldots]}{\mathrm{Tr}[\hat{\rho}(t_0)]}$$

$$\langle \hat{\varphi}(x) \rangle = 0$$

$$\langle \hat{\varphi}(x) \rangle \neq 0 \quad \Rightarrow \quad \hat{\varphi}(x) \rightarrow \hat{\varphi}(x) - \langle \hat{\varphi}(x) \rangle$$

$$\left\langle \tilde{T}\left(\hat{\varphi}(x)\hat{\varphi}(x)\right)\right\rangle$$

Perturbative expansion – path integral approach

Generating functional – 1st step

$$\bar{W}[J,\varphi_0,\varphi_0'] \equiv \int_{BC} \mathcal{D}\varphi(x)\, e^{i\int_C d^4x \mathcal{L}(x)}$$

Boundary condition

$$\varphi(t \to -\infty + i0^+, \mathbf{x}) = \varphi_0(\mathbf{x})$$
$$\varphi(t \to -\infty - i0^+, \mathbf{x}) = \varphi_0'(\mathbf{x})$$

Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_{\text{int}}(x) + J(x)\varphi(x)$$

Generating functional – 2nd step

$$W[J] = N\int \mathcal{D}\varphi_0(\mathbf{x})\mathcal{D}\varphi_0'(\mathbf{x})\,\rho\left[\varphi_0,\varphi_0'\right]\bar{W}[J,\varphi_0,\varphi_0']$$

$$N^{-1} = W[J=0]$$

Green's functions from generating functional

$$i\Delta(x_1, x_2, \ldots x_n) \equiv N \int \mathcal{D}\varphi_0(\mathbf{x})\, \mathcal{D}\varphi_0'(\mathbf{x})\, \rho\left[\varphi_0, \varphi_0'\right]$$

$$\times \int_{BC} \mathcal{D}\varphi(x)\, \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\, e^{i\int_C d^4x \mathcal{L}(x)}$$

$$i\Delta(x_1, x_2, \ldots x_n) = (-i)^n \frac{\delta^n}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} W[J]\Big|_{J=0}$$

Perturbative expansion as in vacuum QFT

Lectures III & IV

From QFT to kinetic theory

- Dyson-Schwinger equations
- Kadanoff-Baym equations
- Transport & mass-shell equations of Green's functions
- Gradient expansion
- Perturbative expansion of self-energy
- Distribution functions
- Kinetic equations

Dyson-Schwniger equation

Contour Green's function

$$\begin{cases} [\Box_x + m^2]\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C dx' \, \Pi(x,x')\,\Delta(x',y) \\ [\Box_y + m^2]\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C dx' \, \Delta(x,x')\,\Pi(x',y) \end{cases}$$

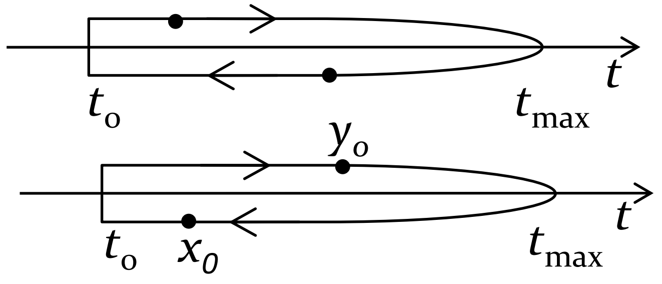
$$\Delta_0{}^{-1}\Delta = 1 - \Pi\Delta \qquad \Delta\Delta_0{}^{-1} = 1 - \Delta\Pi$$

$$\Delta = \Delta_0 - \Delta_0\Pi\Delta \qquad \Delta = \Delta_0 - \Delta\Pi\Delta_0$$

$$[\Box_x + m^2]\Delta_0(x,y) = -\delta_C^{(4)}(x,y)$$

Contour vs. real-time Green's functions

$$\begin{cases} \Delta^{>}(x,y) = \Delta(x,y) \\ \Delta^{<}(x,y) = \Delta(x,y) \end{cases}$$



$$\Delta(x,y) = \theta_C(x,y)\,\Delta^{>}(x,y) + \theta_C(y,x)\,\Delta^{<}(x,y)$$

$$\Pi(x,y) = \delta^{(4)}(x-y)\,\Pi_{\mathrm{MF}}(x) + \theta_C(x,y)\,\Pi^{>}(x,y) + \theta_C(y,x)\,\Pi^{<}(x,y)$$

Kadanoff-Baym equations

$$\left\{\begin{aligned}
[\Box_x + m^2 - \Pi_{\rm MF}(x)]\Delta^>(x,y) &= \int_{-\infty}^{y_0} dx'\,\Pi^>(x,x')\Big(\Delta^>(x',y) - \Delta^<(x',y)\Big) \\
&\quad + \int_{-\infty}^{x_0} dx'\Big(\Pi^>(x,x') - \Pi^<(x,x')\Big)\Delta^>(x',y) \\
[\Box_y + m^2 - \Pi_{\rm MF}(x)]\Delta^>(x,y) &= \int_{-\infty}^{y_0} dx'\,\Delta^>(x,x')\Big(\Pi^<(x',y) - \Pi^>(x',y)\Big) \\
&\quad + \int_{-\infty}^{x_0} dx'\Big(\Delta^>(x,x') - \Delta^<(x,x')\Big)\Pi^>(x',y)
\end{aligned}\right.$$

Analogous equations of $\Delta^<(x,y)$

Retarded & advanced functions

$$\begin{cases} \Delta^{+}(x,y) = \theta(x_0 - y_0)\left(\Delta^{>}(x,y) - \Delta^{<}(x,y)\right) \\ \Delta^{-}(x,y) = -\theta(y_0 - x_0)\left(\Delta^{>}(x,y) - \Delta^{<}(x,y)\right) \end{cases}$$

$$\begin{cases} \Pi^{+}(x,y) = \theta(x_0 - y_0)\left(\Pi^{>}(x,y) - \Pi^{<}(x,y)\right) \\ \Pi^{-}(x,y) = -\theta(y_0 - x_0)\left(\Pi^{>}(x,y) - \Pi^{<}(x,y)\right) \end{cases}$$

Kadanoff-Baym equations

$$\begin{cases} [\Box_x + m^2 - \Pi_{\text{MF}}(x)]\Delta^>(x,y) = \int dx' \left(\Pi^>(x,x')\Delta^-(x',y) + \Pi^+(x,x')\Delta^>(x',y) \right) \\ [\Box_y + m^2 - \Pi_{\text{MF}}(x)]\Delta^>(x,y) = \int dx' \left(\Delta^>(x,x')\Pi^-(x',y) - \Delta^+(x,x')\Pi^>(x',y) \right) \end{cases}$$

$$\begin{cases} [\Box_x + m^2 - \Pi_{\text{MF}}(x)]\Delta^<(x,y) = \int dx' \left(\Pi^+(x,x')\Delta^<(x',y) + \Pi^<(x,x')\Delta^-(x',y) \right) \\ [\Box_y + m^2 - \Pi_{\text{MF}}(x)]\Delta^<(x,y) = \int dx' \left(\Delta^+(x,x')\Pi^<(x',y) - \Delta^<(x,x')\Pi^-(x',y) \right) \end{cases}$$

Gradient expansion

Wigner transformation

$$\Delta(X,p)=\int d^4u\, e^{ipu}\Delta(X+\tfrac{1}{2}u, X-\tfrac{1}{2}u)$$

$\Delta(X+\tfrac{1}{2}u, X-\tfrac{1}{2}u)$ slowly varies in X and is peaked around $u = 0$.

$$\left|\partial_X^\mu \partial_p^\nu \Delta(X,p)\right| << \left|\Delta(X,p)\right|$$

Wigner transformation & gradient expansion up to 1st order

$$\int d^4x' \, f(x,x')g(x',x) \quad \to \quad f(X,p)g(X,p) + \frac{i}{2}\left[\frac{\partial f(X,p)}{\partial p_\mu}\frac{\partial g(X,p)}{\partial X^\mu} - \frac{\partial f(X,p)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu} \right]$$

$$h(x)g(x,y) \quad \to \quad h(X)g(X,p) - \frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}$$

$$h(y)g(x,y) \quad \to \quad h(X)g(X,p) + \frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}$$

$$\partial_x^\mu f(x,y) \quad \to \quad \left(-ip^\mu + \tfrac{1}{2}\partial_X^\mu\right) f(X,p)$$

$$\partial_y^\mu f(x,y) \quad \to \quad \left(ip^\mu + \tfrac{1}{2}\partial_X^\mu\right) f(X,p)$$

From Kadanoff-Baym to transport & mass-shell equations

$$\begin{cases} [\Box_x + m^2 - \Pi_{\mathrm{MF}}(x)]\Delta^>(x,y) = \int dx' \left(\Pi^>(x,x')\Delta^-(x',y) + \Pi^+(x,x')\Delta^>(x',y)\right) \\ [\Box_y + m^2 - \Pi_{\mathrm{MF}}(x)]\Delta^>(x,y) = \int dx' \left(\Delta^>(x,x')\Pi^-(x',y) - \Delta^+(x,x')\Pi^>(x',y)\right) \end{cases}$$

Leading order in gradient expansion

Transport equation

$$\left[p_\mu \partial^\mu - \tfrac{1}{2}\partial_\mu \Pi_{\mathrm{MF}}(X)\partial_p^\mu \right]\Delta^>(X,p) = \tfrac{i}{2}\left[\Pi^<(X,p)\Delta^>(X,p) - \Pi^>(X,p)\Delta^<(X,p)\right]$$

Mass-shell equation

$$\left[\tfrac{1}{4}\partial^2 - p^2 + m^2 - \Pi_{\mathrm{MF}}(X)\right]\Delta^>(X,p) = \tfrac{1}{2}\Big[\Pi^>(X,p)\left(\Delta^+(X,p) + \Delta^-(X,p)\right) + \left(\Pi^+(X,p) + \Pi^-(X,p)\right)\Delta^>(X,p)\Big]$$

Quasi-particle approximation

$$\frac{1}{m^2}\left|\partial^2 \Delta^>(X,p)\right| << \left|\Delta^>(X,p)\right|$$

Mass-shell equation

$$\left[-p^2+m^2-\mathrm{Re}\,\Pi(X,p)\right]\Delta^>(X,p)=\tfrac{1}{2}\Pi^>(X,p)\overbrace{\left(\Delta^+(X,p)+\Delta^-(X,p)\right)}^{\approx 0}$$

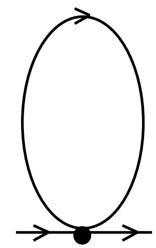
$$\mathrm{Re}\,\Pi(X,p)\equiv\Pi_{\mathrm{MF}}(X)+\tfrac{1}{2}\left(\Pi^+(X,p)+\Pi^-(X,p)\right)$$

Dispersion equation

$$-p^2+m^2-\mathrm{Re}\,\Pi(X,p)=0$$

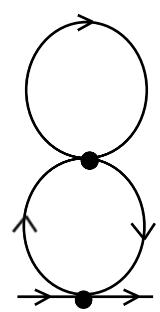
Perturbative expansion of Π

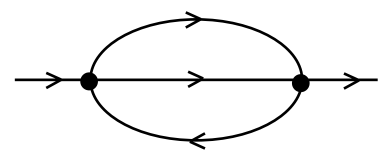
g



$$\mathcal{L}_{\text{int}} = -\frac{g}{2!2!}(\varphi\varphi^\dagger)^2$$

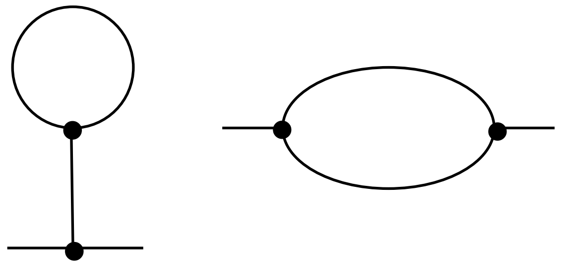
g^2





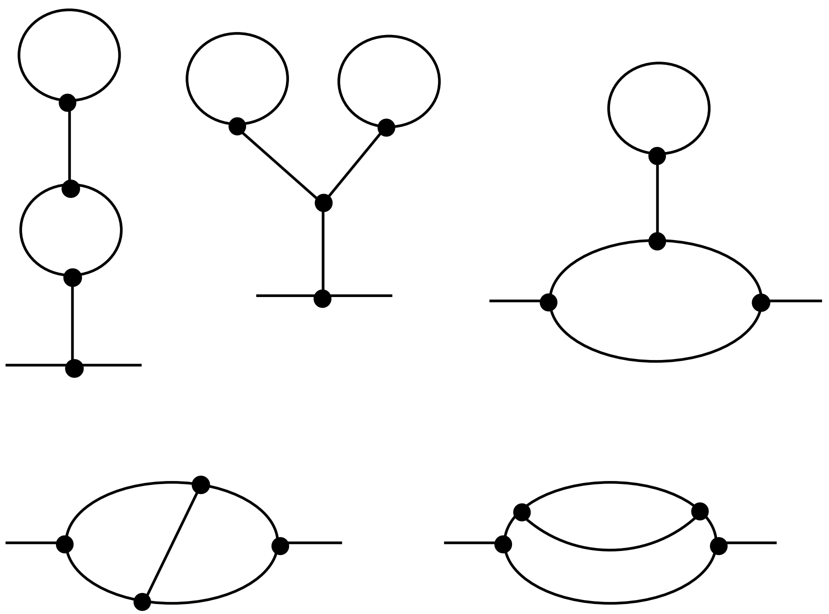
Perturbative expansion of Π

g^2



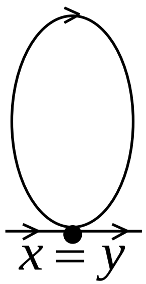
$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$

g^4

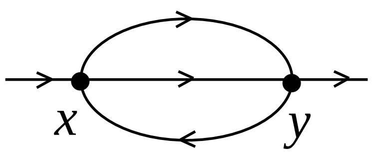


Perturbative expansion of Π

$$\mathcal{L}_{\text{int}} = -\frac{g}{2!2!}(\varphi\varphi^\dagger)^2$$



$$i\Pi(x,y) = \delta^{(4)}(x-y)\underbrace{(-ig)i\Delta(x,x)}_{\Pi_{\text{MF}}(x) = g\Delta^<(x,x)}$$



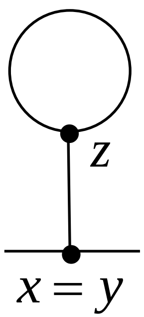
$$i\Pi(x,y) = \tfrac{1}{2}(-ig)^2 i\Delta(x,y)\, i\Delta(x,y)\, i\Delta(y,x)$$

$$\Pi^>(x,y) = \tfrac{1}{2}g^2\Delta^>(x,y)\,\Delta^>(x,y)\,\Delta^<(y,x)$$

$$\Pi^<(x,y) = \tfrac{1}{2}g^2\Delta^<(x,y)\,\Delta^<(x,y)\,\Delta^>(y,x)$$

Perturbative expansion of Π

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$



$$i\Pi(x,y) = \delta^{(4)}(x-y)\,\tfrac{1}{2}(-ig)^2 \int_C d^4z\, i\Delta(x,z)\, i\Delta^<(z,z)$$

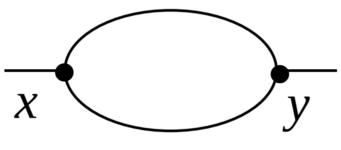
$$= \delta^{(4)}(x-y)\,\tfrac{1}{2}\, g \int d^4z \Big[\overbrace{\Delta^c(x,z) - \Delta^<(x,z)}^{\Delta^+(x,z)} \Big] \Delta^<(z,z)$$

$$= \delta^{(4)}(x-y)\,\underbrace{\tfrac{1}{2}\, g \int d^4z\, \Delta^+(x,z)\Delta^<(z,z)}$$

$$\Pi_{\text{MF}}(x) = \tfrac{1}{2} g^2 \int d^4z\, \Delta^+(x,z)\Delta^<(z,z)$$

Perturbative expansion of Π

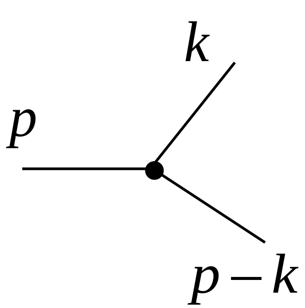
$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$



$$i\Pi(x,y) = \tfrac{1}{2}(-ig)^2 i\Delta(x,y)\, i\Delta(y,x)$$

$$\Pi^>(x,y) = -\tfrac{i}{2} g^2 \Delta^>(x,y)\,\Delta^<(y,x)$$

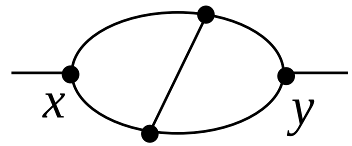
$$\Pi^<(x,y) = -\tfrac{i}{2} g^2 \Delta^<(x,y)\,\Delta^>(y,x)$$



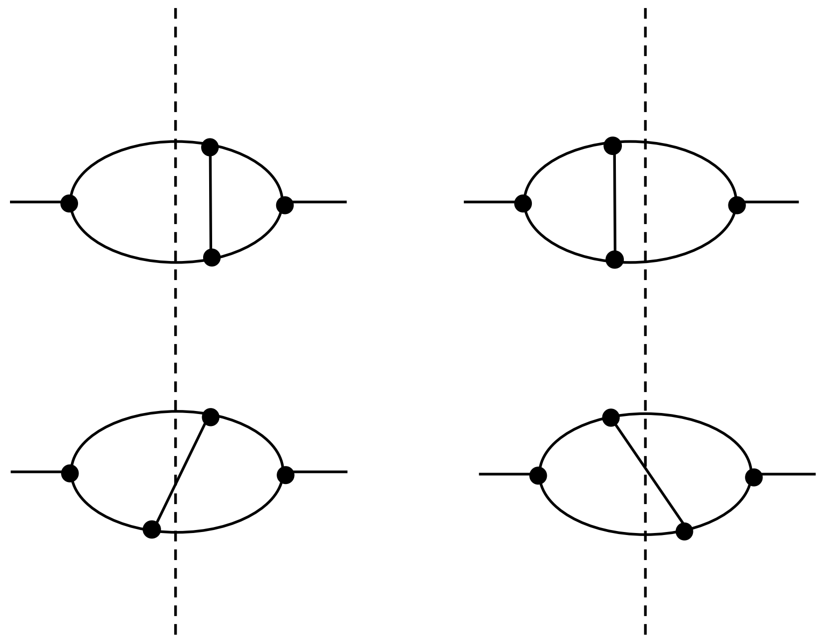
If $p^2 = k^2 = (p-k)^2 = m^2$ kinematically not allowed.

Perturbative expansion of Π

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$

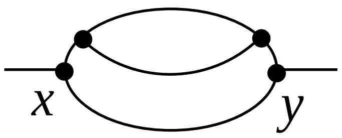


$$\Pi^{>}(x,y)$$

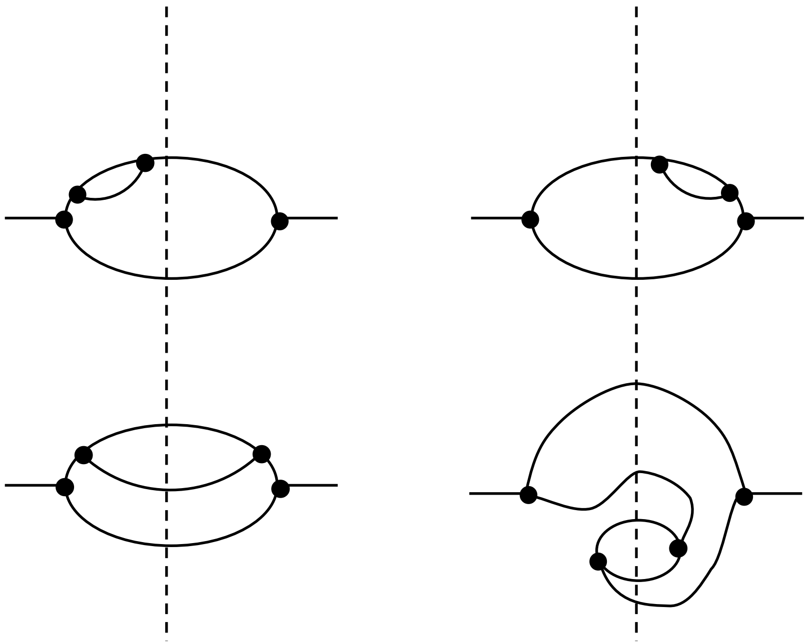


Perturbative expansion of Π

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$

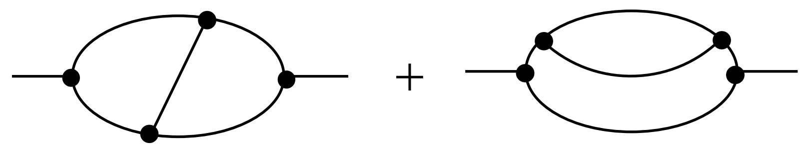


$\Pi^{>}(x,y)$

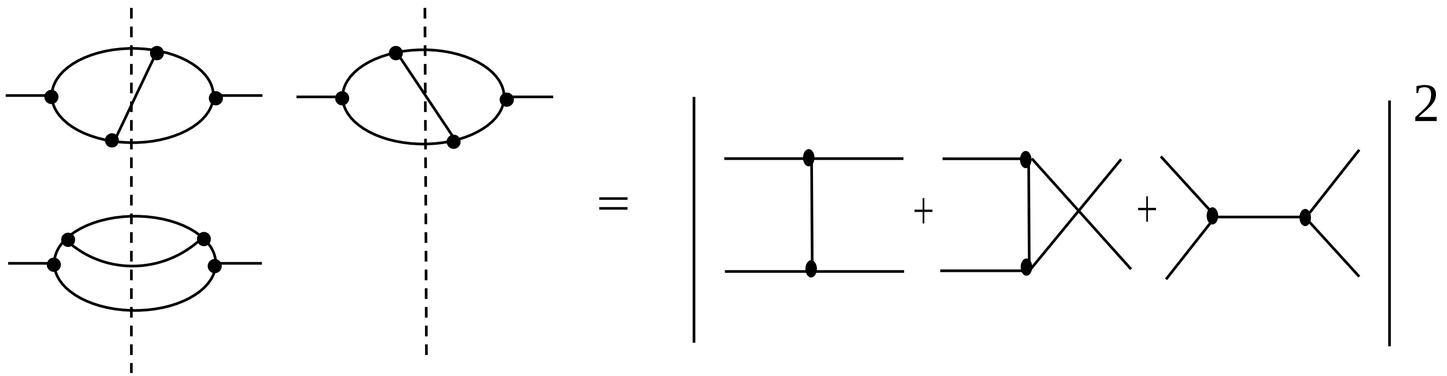


Perturbative expansion of Π

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$



$$\Pi^{>}(x,y)$$



Effective mass

Mass-shell equation

$$\left[-p^2 + m^2 - \mathrm{Re}\,\Pi(X,p)\right]\Delta^{>}(X,p) \approx 0$$

$$\mathrm{Re}\,\Pi(X,p) \equiv \Pi_{\mathrm{MF}}(X) + \tfrac{1}{2}\left(\Pi^{+}(X,p) + \Pi^{-}(X,p)\right)$$

Dispersion equation

$$-p^2 + m^2 - \mathrm{Re}\,\Pi(X,p) = 0$$

Mass-shell equation in leading order in *g*

$$\left[p^2 - m^2 + \Pi_{\mathrm{MF}}(X)\right]\Delta^{>}(X,p) = 0$$

$$m_*^2(X) \equiv m^2 - \Pi_{\mathrm{MF}}(X)$$

Distribution function

$$\begin{cases} \theta(p_0)\, i\Delta^{<}(X,p) \equiv \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 - E_{\mathbf{p}})\, f(X,\mathbf{p}) \\ \theta(-p_0)\, i\Delta^{>}(X,p) \equiv \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 + E_{\mathbf{p}})\, \bar{f}(X,-\mathbf{p}) \end{cases}$$

$$E_{\mathbf{p}} \equiv \sqrt{m_*^2(X) + \mathbf{p}^2}$$

$$\Delta^{>}(X,p) - \Delta^{<}(X,p) = \Delta^{+}(X,p) - \Delta^{-}(X,p)$$

$$\begin{cases} i\Delta^{<}(X,p) = \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 - E_{\mathbf{p}})\, f(X,\mathbf{p}) + \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 + E_{\mathbf{p}})\left[\bar{f}(X,-\mathbf{p}) + 1\right] \\ i\Delta^{<}(X,p) = \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 - E_{\mathbf{p}})\left[f(X,\mathbf{p}) + 1\right] + \dfrac{\pi}{E_{\mathbf{p}}}\delta(p_0 + E_{\mathbf{p}})\, \bar{f}(X,-\mathbf{p}) \end{cases}$$

Current

$$j^{\mu}(X) = 2\int \frac{d^4p}{(2\pi)^4}\, p^{\mu} i\Delta^{>}(X,p) = -2\int \frac{d^3p}{(2\pi)^3}\frac{p^{\mu}}{E_{\mathbf{p}}}\left[f(X,\mathbf{p}) - \bar{f}(X,\mathbf{p}) + 1 \right]$$

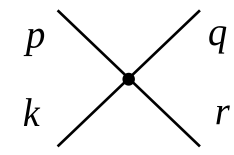
Vacuum contribution needs to be subtracted

$$j^{\mu}(X) = -2\int \frac{d^3p}{(2\pi)^3}\frac{p^{\mu}}{E_{\mathbf{p}}}\left[f(X,\mathbf{p}) - \bar{f}(X,\mathbf{p}) \right]$$

Transport equation

$$\left[p_\mu \partial^\mu - \tfrac{1}{2}\partial_\mu \Pi_{\rm MF}(X)\partial_p^\mu \right]\Delta^>(X,p) = \tfrac{i}{2}\left[\Pi^<(X,p)\Delta^>(X,p) - \Pi^>(X,p)\Delta^<(X,p)\right]$$

$$\mathcal{L}_{\rm int} = -\frac{g}{2!2!}(\varphi\varphi^\dagger)^2$$



$$\left[p_\mu \partial^\mu - F_\mu(X)\partial_p^\mu \right] f(X,\mathbf{p})$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}} \frac{d^3r}{(2\pi)^3 2E_{\mathbf{r}}} (2\pi)^4 \delta^{(4)}(p+k-q-r)$$

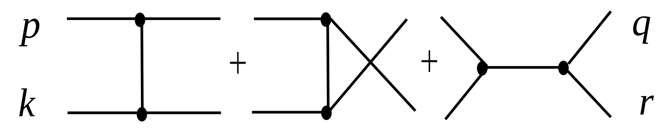
$$\times \Big\{ \tfrac{1}{2}|M_1|^2 \left[f_p f_k \left(f_q+1\right)\left(f_r+1\right) - f_q f_r \left(f_p+1\right)\left(f_q+1\right)\right]$$

$$+ |M_2|^2 \left[f_p \bar{f}_k \left(f_q+1\right)\left(\bar{f}_r+1\right) - f_q \bar{f}_r \left(f_p+1\right)\left(\bar{f}_q+1\right)\right]\Big\}$$

$$F^\mu(X) = -\tfrac{1}{2} g \partial^\mu \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[f(X,\mathbf{p}) + \bar{f}(X,\mathbf{p})\right] \qquad f_p \equiv f(X,\mathbf{p}), \quad M_1 = M_2 = -ig$$

Transport equation

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$



$$\left[p_\mu \partial^\mu - F_\mu(X)\partial_p^\mu \right] f(X,\mathbf{p})$$
$$= \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}} \frac{d^3r}{(2\pi)^3 2E_{\mathbf{r}}} (2\pi)^4 \delta^{(4)}(p+k-q-r)$$
$$\times |M|^2 \left[f_p f_k \left(f_q+1\right)\left(f_r+1\right) - f_q f_r \left(f_p+1\right)\left(f_q+1\right)\right]$$

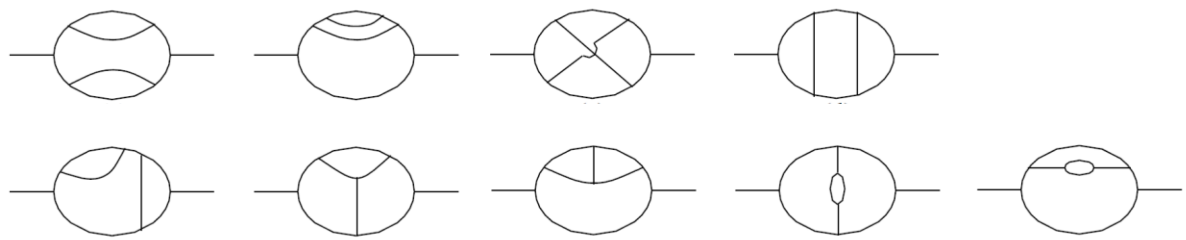
$$F^\mu(X) = -g\partial^\mu \int d^4X' \Delta^+(X,X') \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} f(X',\mathbf{p})$$

$$f_p \equiv f(X,\mathbf{p}), \qquad M = ig^2\left[\Delta^c(X,q-p) + \Delta^c(X,r-p) + \Delta^c(X,p+k)\right]$$

Higher order contributions

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\varphi^3$$

g^6



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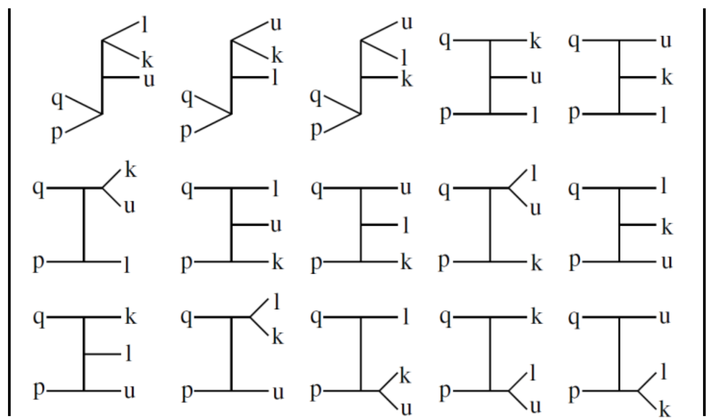
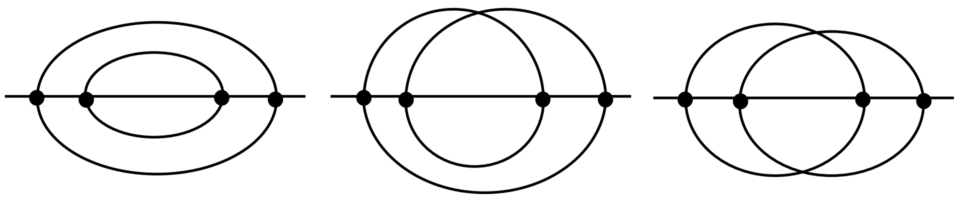


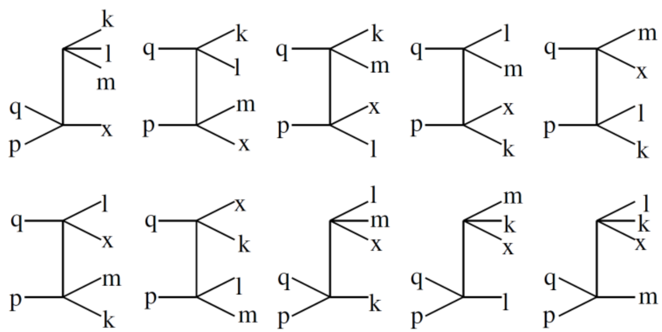
2

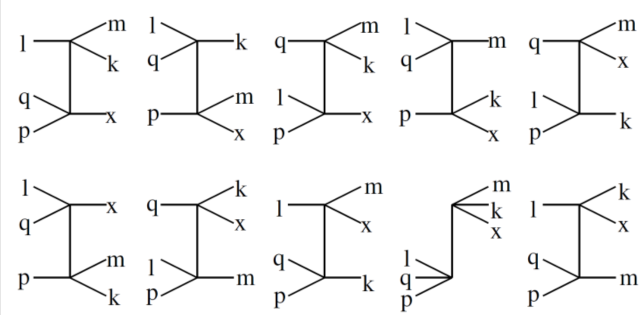
$2 \leftrightarrow 3$

Higher order contributions

$$\mathcal{L}_{\text{int}} = -\frac{g}{2!2!}(\varphi\varphi^{\dagger})^2$$

g^4 

$= \Big|$ 

$\Big|^2 + \Big|$ 

$\Big|^2$

$2 \leftrightarrow 4$ $\qquad$ $3 \leftrightarrow 3$

Conclusions

The Keldysh-Schwinger formalism is an efficient tool to describe relativistic and non-relativistic equilibrium and non-equilibrium statistical systems.

In particular, the formalism allows one to systematically derive transport theory from underlying quantum field dynamics.

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