Gluon Collective Modes in Anisotropic Quark-Gluon Plasma

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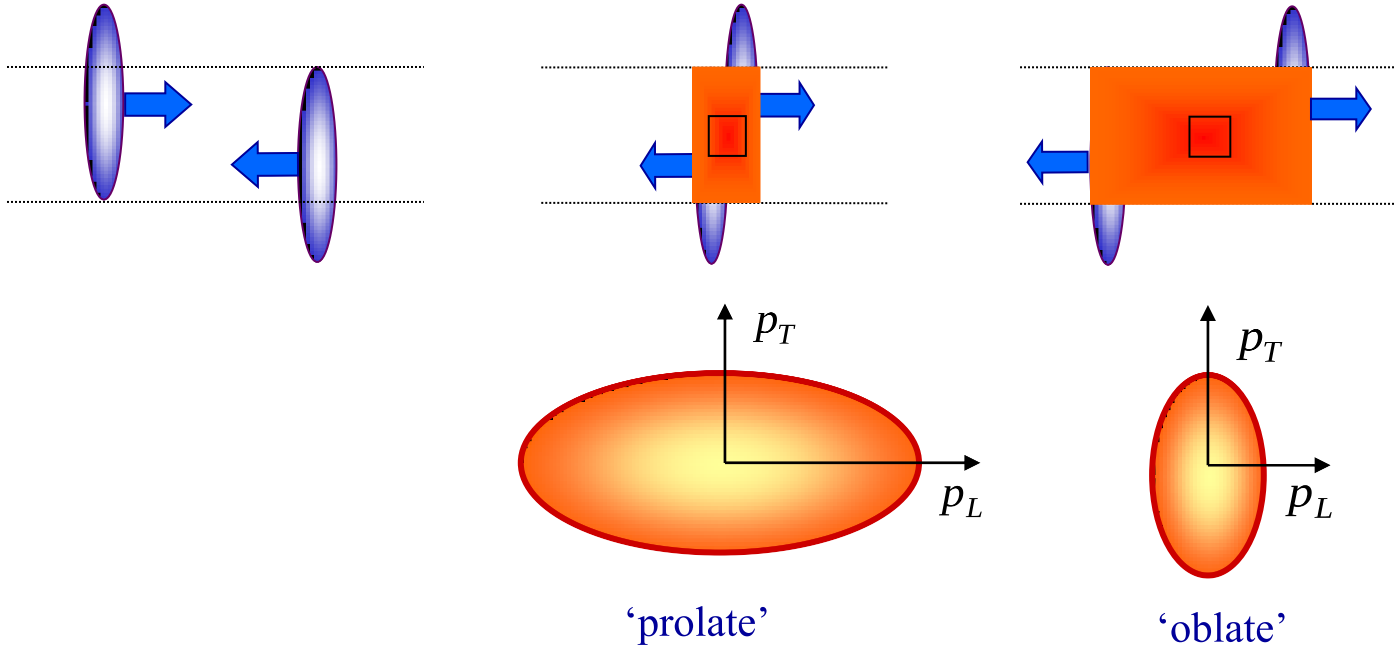
Motivation

- Spectrum of collective excitations is an important characteristics of any many body system.

- Anisotropic plasma is qualitatively different than the isotropic one.

Motivation cont.

- QGP from relativistic heavy-ion collisions is anisotropic.



Motivation cont.

- Existing analyses of collective excitations

St. Mrówczyński, Phys. Lett. B **314**, 118 (1993), Phys. Rev. C **49**, 2191 (1994)

P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003), *ibid* D **70**, 116006 (2004)

P.B. Arnold, J.Lenaghan and G.D. Moore, JHEP **0308**, 002 (2003) …..

……

are not complete.

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon polarization tensor

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Gluon collective modes – solutions $\omega(\mathbf{k})$

Instabilities – solutions with $\mathrm{Im}\omega > 0$ $\Rightarrow$ $A^\mu(x) \sim e^{\mathrm{Im}\omega t}$

Dynamical information is hidden in the polarization tensor $\Pi^{\mu\nu}(k)$

Polarization tensor

Diagrammatic hard-loop approach

Linear response analysis within kinetic theory

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\Big[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\Big]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

$f(\mathbf{p})$ - momentum distribution of plasma constituents

Momentum distribution

Isotropic momentum distribution rescaled in the direction of **n**

$$f_\xi(\mathbf{p}) = C_\xi \, f_{\text{iso}}\left(\sqrt{\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2}\right)$$

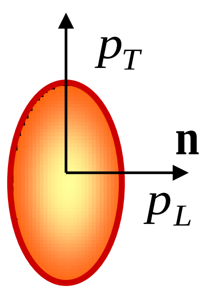
$\xi \in (-1, \infty)$

$$f_\sigma(\mathbf{p}) = C_\sigma \, f_{\text{iso}}\left(\sqrt{(\sigma+1)\mathbf{p}^2 - \sigma(\mathbf{p}\cdot\mathbf{n})^2}\right)$$

$\sigma \in (-1, \infty)$

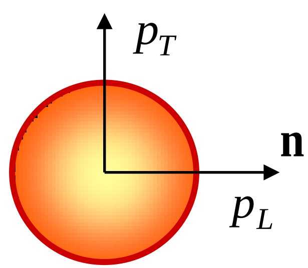
$$f_\xi(\mathbf{p}) \xrightarrow[\xi\to\infty]{} \delta(p_L)$$

$$f_\sigma(\mathbf{p}) \xrightarrow[\sigma\to\infty]{} \delta(p_T)$$



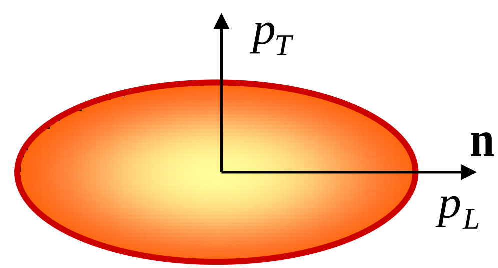
oblate

$\xi > 0, \quad \sigma < 0$



isotropic

$\xi = \sigma = 0$



prolate

$\xi < 0, \quad \sigma > 0$

P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003)

Choosing normalization constants

$$f_\xi(\mathbf{p}) = C_\xi\ f_{\text{iso}}\left(\sqrt{\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2}\right)$$

$$f_\sigma(\mathbf{p}) = C_\sigma\ f_{\text{iso}}\left(\sqrt{(\sigma+1)\mathbf{p}^2 - \sigma(\mathbf{p}\cdot\mathbf{n})^2}\right)$$

Parton number normalization

$$\int \frac{d^3p}{(2\pi)^3} f_\xi(\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3} f_{\text{iso}}(|\mathbf{p}|) = \int \frac{d^3p}{(2\pi)^3} f_\sigma(\mathbf{p}) \quad \Rightarrow \quad \begin{cases} C_\xi = \sqrt{1+\xi} \\ C_\sigma = \sigma + 1 \end{cases}$$

Debye mass normalization

$$m^2 \equiv \int \frac{d^3p}{(2\pi)^3} \frac{f_\xi(\mathbf{p})}{|\mathbf{p}|} = \int \frac{d^3p}{(2\pi)^3} \frac{f_{\text{iso}}(|\mathbf{p}|)}{|\mathbf{p}|} = \int \frac{d^3p}{(2\pi)^3} \frac{f_\sigma(\mathbf{p})}{|\mathbf{p}|} \quad \Rightarrow \quad \begin{cases} C_\xi = \dfrac{\sqrt{\xi}}{\text{Arctanh}\sqrt{\xi}} \\ C_\sigma = \dfrac{\sqrt{\sigma(\sigma+1)}}{\text{Arctan}\sqrt{\dfrac{\sigma}{\sigma+1}}} \end{cases}$$

Dispersion equation

Dispersion equation: $$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

4 x 4 matrix

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Dielectric tensor: $$\varepsilon^{ij}(\omega,\mathbf{k}) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(\omega,\mathbf{k})$$ 3 x 3 matrix

Dispersion equation: $$\det\left[\Sigma(\omega,\mathbf{k})\right] = 0$$

$$\Sigma^{ij}(\omega,\mathbf{k}) \equiv (\omega^2 - \mathbf{k}^2)\delta^{ij} + k^i k^j - \Pi^{ij}(\omega,\mathbf{k})$$ inverse gluon propagator in temporal axial gauge

Solution $\omega(\mathbf{k})$ - gluon collective mode

Instead of looking for zeros of $\det\Sigma$, one looks for poles of Σ^{-1}

How to invert matrix Σ ?

Method to inverse the matrix Σ

The symmetric matrix Σ depends on **k** and **n**

$$\Sigma = \alpha A + \beta B + \gamma C + \delta D$$

basis of 4 projectors

$$\begin{cases} A^{ij} = \delta^{ij} - \dfrac{k^i k^j}{\mathbf{k}^2}, & B^{ij} = \dfrac{k^i k^j}{\mathbf{k}^2} \\ C^{ij} = \dfrac{n_T^i n_T^j}{\mathbf{n}_T^2}, & D^{ij} = n_T^i k^j + k^i n_T^j \end{cases}$$

$$n_T^i \equiv \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) n^j$$

$$k^i \Sigma^{ij} k^j = k^2 \beta, \qquad n_T^i \Sigma^{ij} k^j = n_T^2 k^2 \delta,$$

$$n_T^i \Sigma^{ij} n_T^j = n_T^2 (\alpha + \gamma), \qquad \mathrm{Tr}\,\Sigma = 2\alpha + \beta + \gamma,$$

$\Rightarrow$ $\alpha, \beta, \gamma, \delta$

$$\Sigma^{-1} = \bar{\alpha} A + \bar{\beta} B + \bar{\gamma} C + \bar{\delta} D$$

$$\Sigma \Sigma^{-1} = \mathbf{1} \quad \Rightarrow \quad \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$$

P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003)

Dispersion equations

$$\Sigma^{-1} = \frac{A-C}{\omega^2 - k^2 - \alpha} + \frac{(\omega^2 - k^2 - \alpha - \gamma)B + (\omega^2 - \beta)C + \delta D}{(\omega^2 - \beta)(\omega^2 - k^2 - \alpha - \gamma) - k^2 n_T^2 \delta^2}$$

Dispersion equations (poles of Σ^{-1})

1) $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$

2) $(\omega^2 - \beta(\omega, \mathbf{k}))(\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k})) - k^2 n_T^2 \delta^2(\omega, \mathbf{k}) = 0$

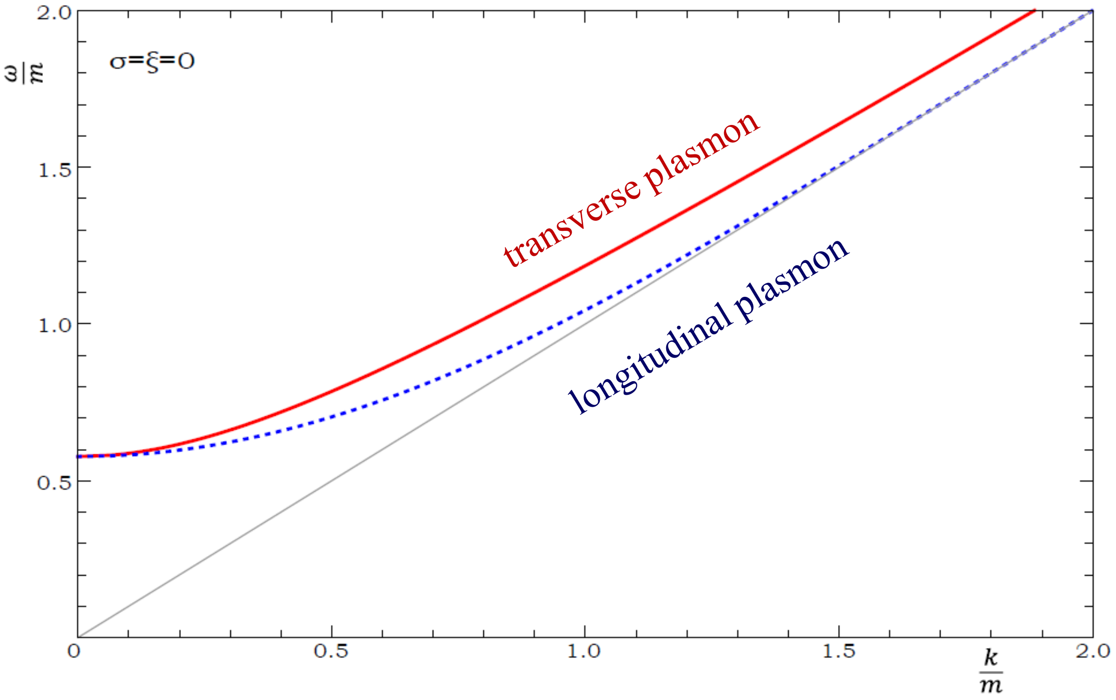
A modes - solutions of the first equation

G modes - solutions of the second equation

Collective modes in isotropic QGP

$$\Sigma = (\omega^2 - k^2 - \alpha_{\text{iso}})A + (\omega^2 - \beta_{\text{iso}})B \qquad (\gamma_{\text{iso}} = \delta_{\text{iso}} = 0)$$

$$\begin{cases} \alpha_{\text{iso}}(\omega,\mathbf{k}) = \omega^2 - k^2 - \dfrac{m^2\omega^2}{2k^2}\left[1 - \left(\dfrac{\omega}{2k} - \dfrac{k}{2\omega}\right)\ln\left(\dfrac{\omega+k}{\omega-k}\right)\right] \\ \beta_{\text{iso}}(\omega,\mathbf{k}) = \omega^2 + \dfrac{m^2\omega^2}{k^2}\left[1 - \dfrac{\omega}{2k}\ln\left(\dfrac{\omega+k}{\omega-k}\right)\right] \end{cases}$$



Dispersion equations

$$\begin{cases} \omega^2 - k^2 - \alpha(\omega,\mathbf{k}) = 0 \\ \omega^2 - \beta(\omega,\mathbf{k}) = 0 \end{cases}$$

4 real solutions
(2 positive & 2 negative)

Debye mass

$$m^2 = g^2 \int \frac{d^3p}{(2\pi)^3} \frac{f_\xi(\mathbf{p})}{|\mathbf{p}|}$$

Transverse and longitudinal modes

- longitudinal modes – $\mathbf{k} \parallel \mathbf{E}, \;\; \delta\rho \sim \mathrm{e}^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$

- transverse modes – $\mathbf{k} \perp \mathbf{E}, \;\; \delta\mathbf{j} \sim \mathrm{e}^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$

E – electric field, **k** – wave vector, ρ – charge density, **j** - current

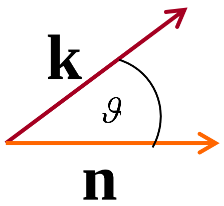
Weakly anisotropic system

$|\xi| << 1$

$$f_\xi(\mathbf{p}) \approx \left(1 + \frac{\xi}{3}\right) f_{\text{iso}}(p) + \frac{\xi}{2} \frac{d\, f_{\text{iso}}(p)}{d\, p} p(\mathbf{v}\cdot\mathbf{n})^2$$

$$\begin{cases} \alpha(\omega,\mathbf{k}) = \left(1+\frac{\xi}{3}\right)\alpha_{iso}(\omega,\mathbf{k}) + \xi \frac{m^2}{8}\left\{\frac{8}{3}\cos^2\vartheta + \frac{2}{3}\left(5 - 19\cos^2\vartheta\right)\frac{\omega^2}{k^2} - 2\left(1 - 5\cos^2\vartheta\right)\frac{\omega^4}{k^4} \right. \\ \left. \quad + \left[1 - 3\cos^2\vartheta - \left(2 - 8\cos^2\vartheta\right)\frac{\omega^2}{k^2} + \left(1 - 5\cos^2\vartheta\right)\frac{\omega^4}{k^4}\right]\frac{\omega}{k}\ln\left(\frac{\omega+k}{\omega-k}\right)\right\}, \\ \\ \beta(\omega,\mathbf{k}) = \ldots, \\ \gamma(\omega,\mathbf{k}) = \ldots, \\ \delta(\omega,\mathbf{k}) = \ldots \end{cases}$$

analytic expressions



Weakly anisotropic system

Dispersion equations

1) $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$

2) $\left(\omega^2 - \beta(\omega, \mathbf{k})\right)\left(\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k})\right) - k^2 n_T^2 \delta^2(\omega, \mathbf{k}) = 0$

$\Downarrow$

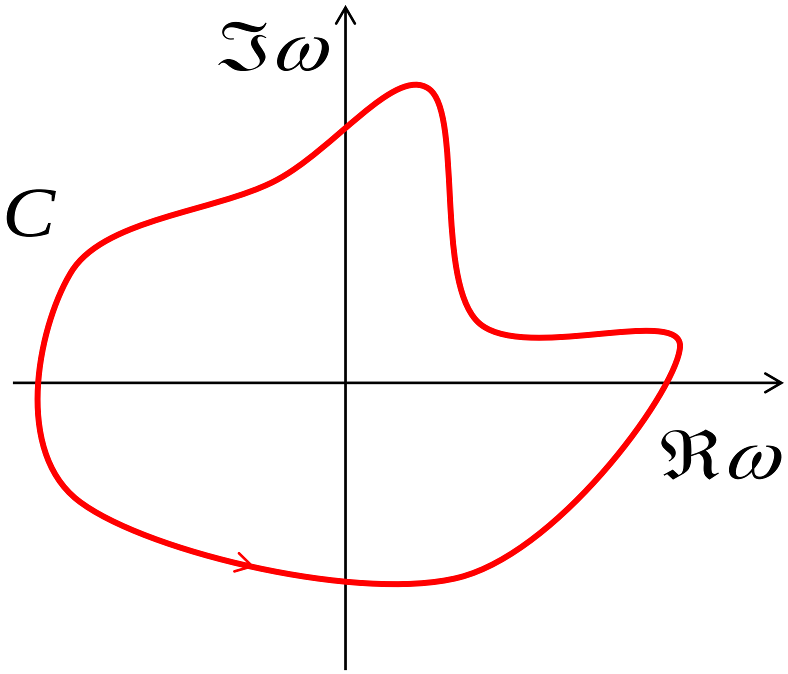
$\delta^2 = O(\xi^2)$

1) $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$

2) $\omega^2 - \beta(\omega, \mathbf{k}) = 0$

3) $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k}) = 0$

Nyquist analysis



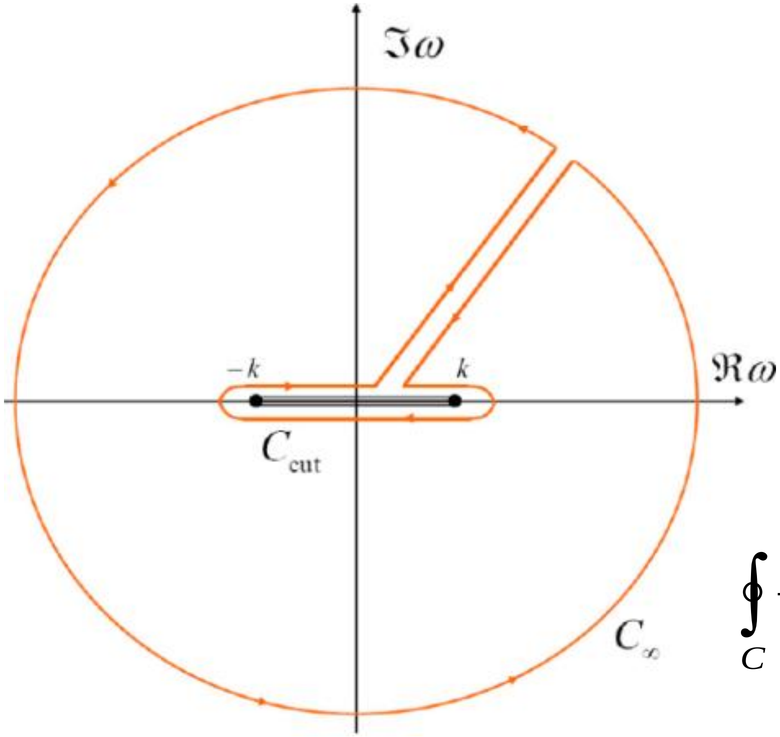
$$f(\omega) = 0$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = n_Z - n_P$$

n_Z - number of zeros of $f(\omega)$ inside C

n_P - number of poles of $f(\omega)$ inside C

Nyquist analysis



$$f(\omega) = 0$$

$$\oint_C \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)} = n_Z - n_P$$

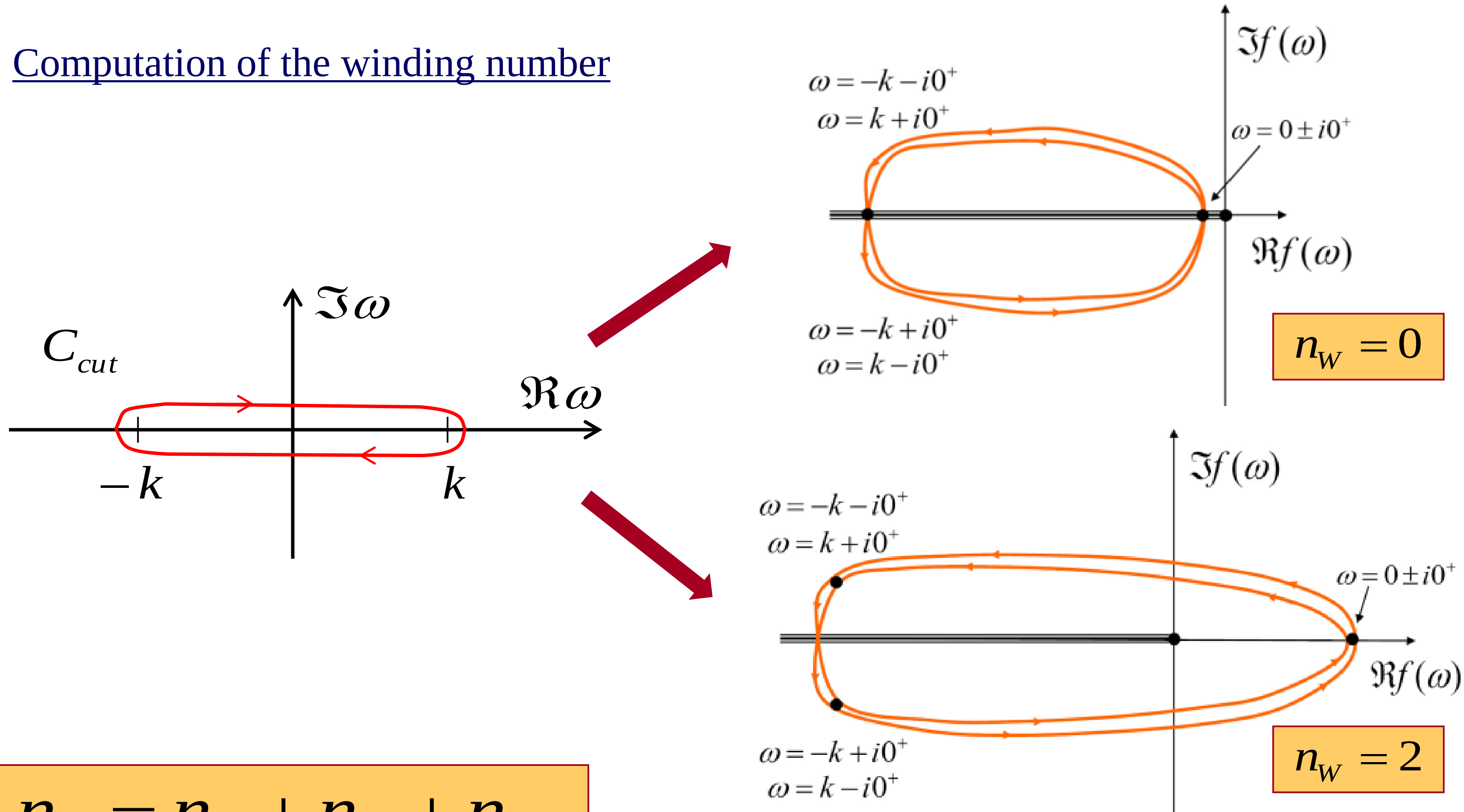
$$\oint_C \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)} = \oint_{C_\infty} \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)} + \oint_{C_{\rm cut}} \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)}$$

C_∞ $$\oint_{C_\infty} \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)} = \lim_{|\omega|\to\infty} \omega\frac{f'(\omega)}{f(\omega)} \equiv n_\infty$$

n_W - winding number

$C_{\rm cut}$ $$\oint_{C_{cut}} \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)} = \frac{1}{2\pi i}\oint_{C_{cut}} \frac{d}{d\omega}\ln f(\omega) = \frac{1}{2\pi i}(\ln f(\omega_e) - \ln f(\omega_s)) \equiv n_W$$

Computation of the winding number



$n_W = 0$

$n_W = 2$

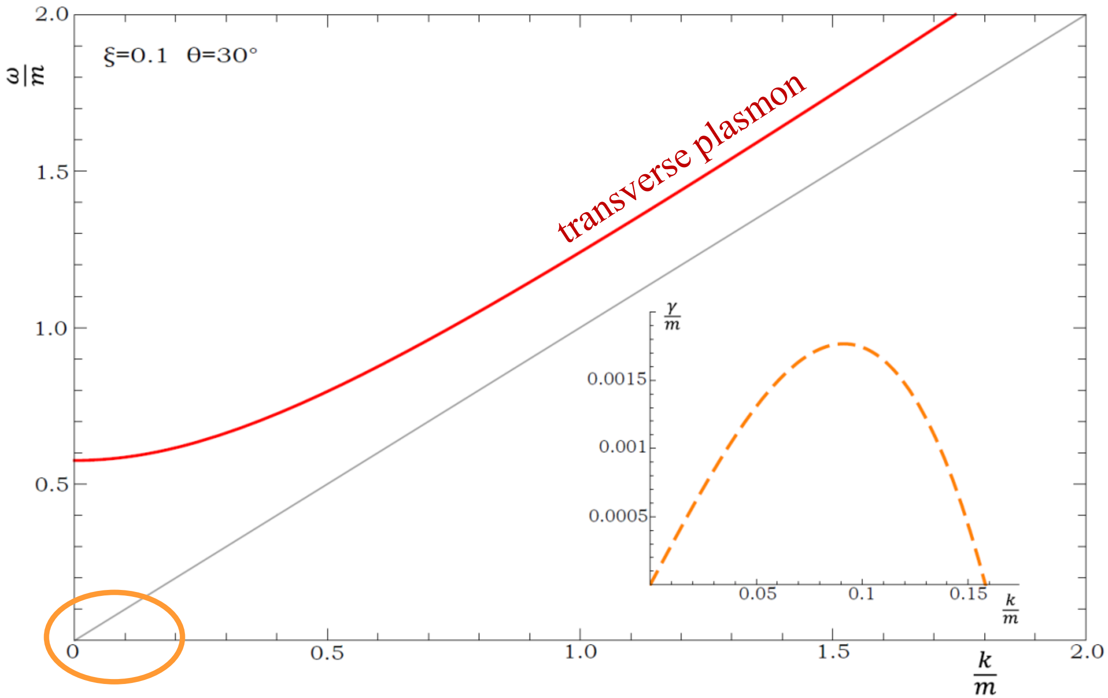
$$n_Z = n_P + n_\infty + n_W$$

Equation $\omega^2 - k^2 - \alpha(\omega,\mathbf{k}) = 0$

$$k^2 - \xi \frac{m^2}{3}\cos^2\vartheta \geq 0$$ 2 solutions

$$k^2 - \xi \frac{m^2}{3}\cos^2\vartheta < 0$$ 4 solutions

$$\omega^2(\mathbf{k}) \underset{k<<m}{\approx} \frac{m^2}{3}\left(1 - \frac{\xi}{15}\right) + \frac{6}{5}\left[1 + \frac{\xi}{14}\left(\frac{4}{15} + \cos^2\vartheta\right)\right]k^2$$



$$\omega(\mathbf{k}) \equiv \pm i\gamma(\mathbf{k}), \quad \gamma \in R$$

$$\gamma(\mathbf{k}) \underset{k>>\gamma}{\approx} \frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2} + 4\left(k_A^2 - k^2\right)} - \frac{\lambda}{k}\right)$$

$$\lambda \equiv \frac{\pi}{4}\left[1 - \frac{\xi}{2}\left(\frac{1}{3} - 3\cos^2\vartheta\right)\right]m^2$$

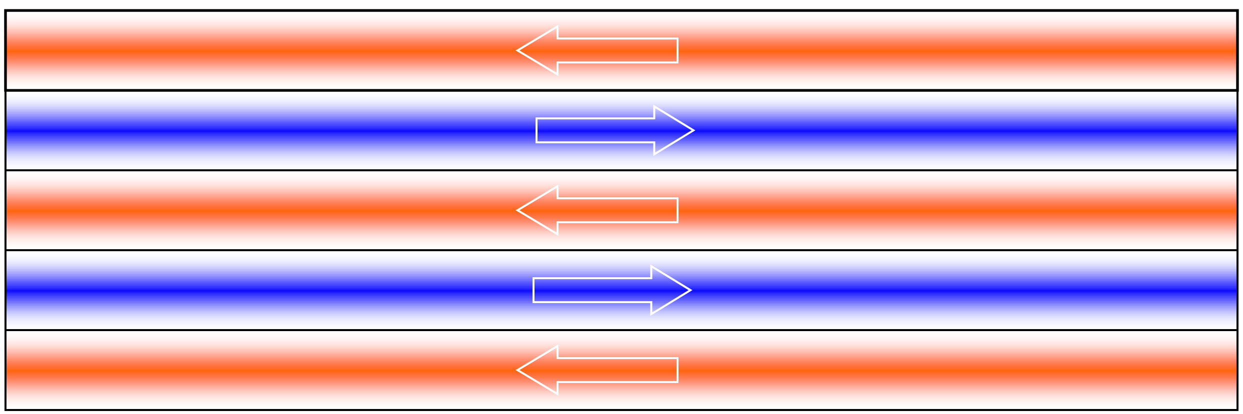
$$k_A \equiv \sqrt{\frac{\xi}{3}}\left|\cos\vartheta\right|$$

Seeds of instability

$\left\langle j_a^{\mu}(x)\right\rangle = 0$ but current fluctuations are finite

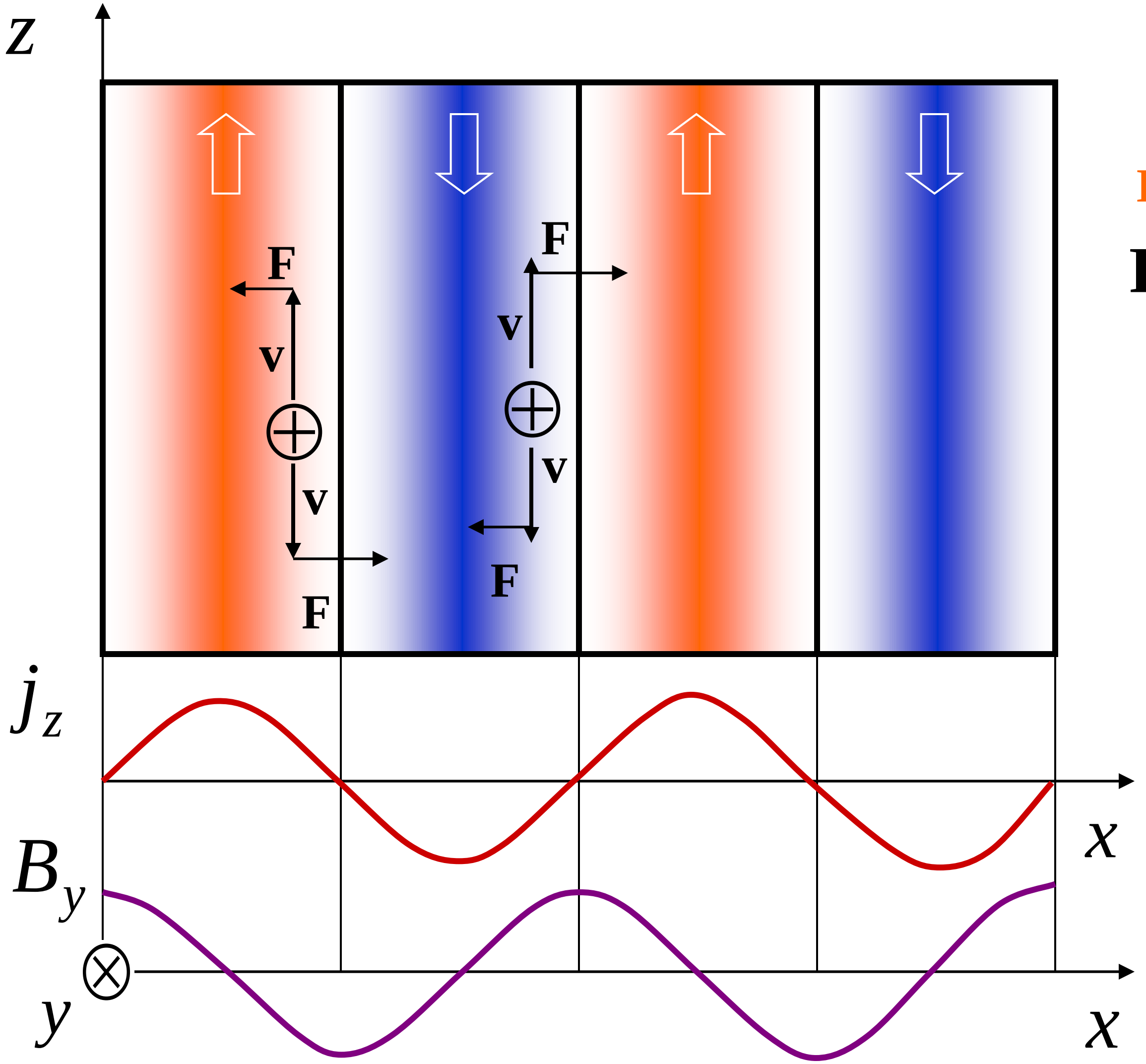
$$\left\langle j_a^{\mu}(x_1)\, j_b^{\nu}(x_2)\right\rangle = \frac{1}{2}\delta^{ab}\int \frac{d^3p}{(2\pi)^3}\frac{p^{\mu}p^{\nu}}{E_p^2}\, f(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{v}t) \neq 0$$

$$x_1 = (t_1, \mathbf{x}_1)\,, \quad x_2 = (t_2, \mathbf{x}_2)\,, \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus

Mechanism of filamentation



Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

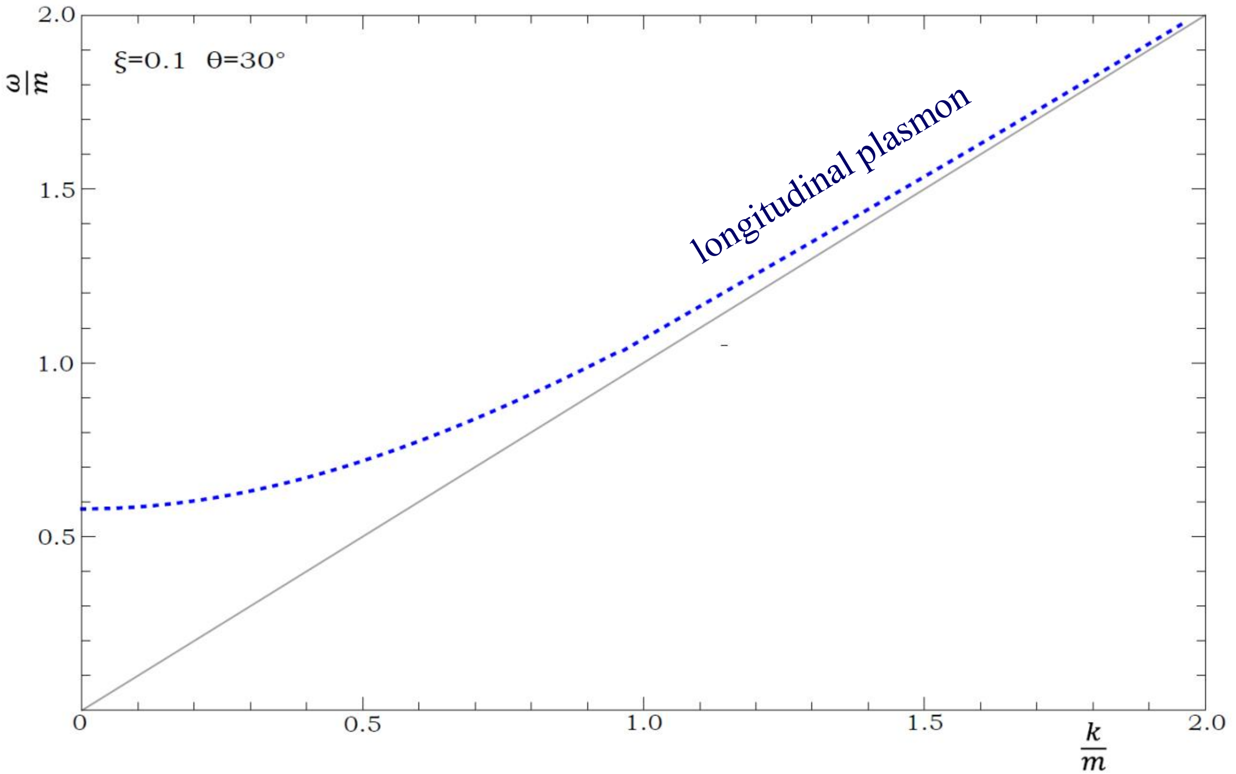
Ampere's law

$$\nabla \times \mathbf{B} = \mathbf{j}$$

Equation $\omega^2 - \beta(\omega,\mathbf{k}) = 0$

2 solutions

$$\omega^2(\mathbf{k}) \underset{k<<m}{\approx} \frac{m^2}{3}\left(1+\frac{\xi}{5}\left(-\frac{1}{3}+\cos^2\vartheta\right)\right)+\frac{3}{5}\left[1+\frac{4\xi}{35}\left(1-3\cos^2\vartheta\right)\right]k^2$$



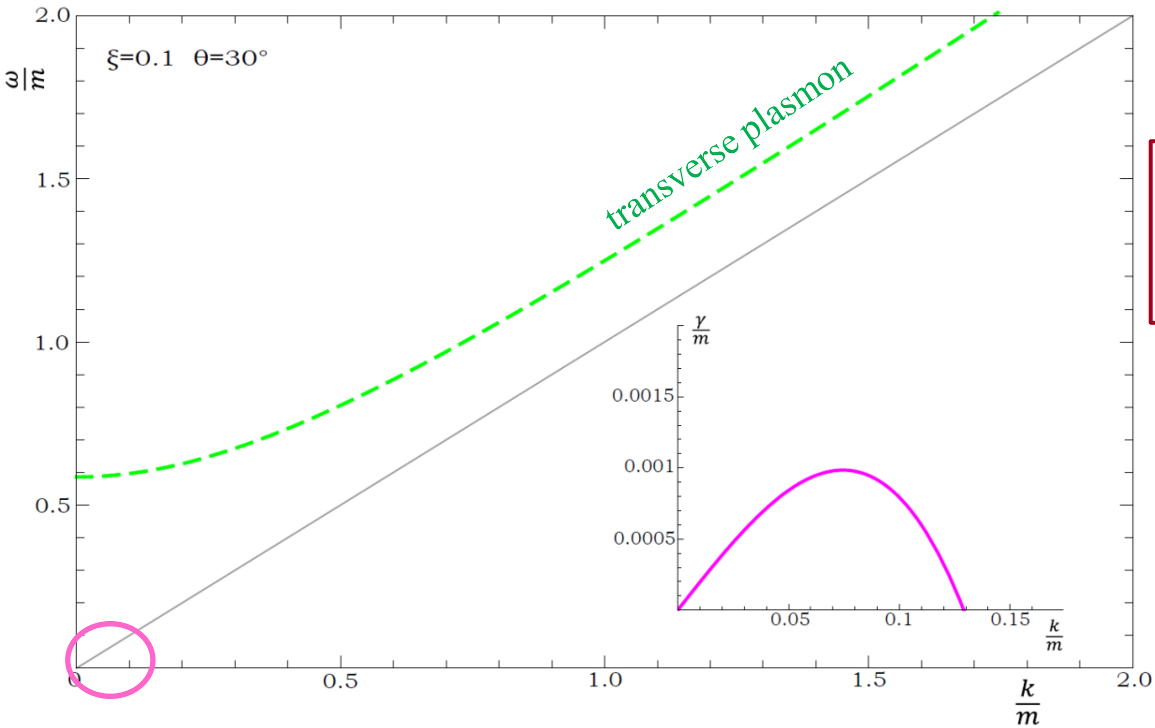
No unstable modes!

Equation $\omega^2 - k^2 - \alpha(\omega,\mathbf{k}) - \gamma(\omega,\mathbf{k}) = 0$

$k^2 + \xi \frac{m^2}{3}\left(1 - 2\cos^2\vartheta\right) \geq 0$ 2 solutions

$k^2 + \xi \frac{m^2}{3}\left(1 - 2\cos^2\vartheta\right) < 0$ 4 solutions

$$\omega^2(\mathbf{k}) \underset{k<<m}{\approx} \frac{m^2}{3}\left(1 + \frac{\xi}{5}\left(\frac{2}{3} - \cos^2\vartheta\right)\right) + \frac{6}{5}\left[1 + \frac{\xi}{5}\left(\frac{23}{42} - \cos^2\vartheta\right)\right]k^2$$



$$\omega(\mathbf{k}) \equiv \pm i\gamma(\mathbf{k}), \quad \gamma \in R$$

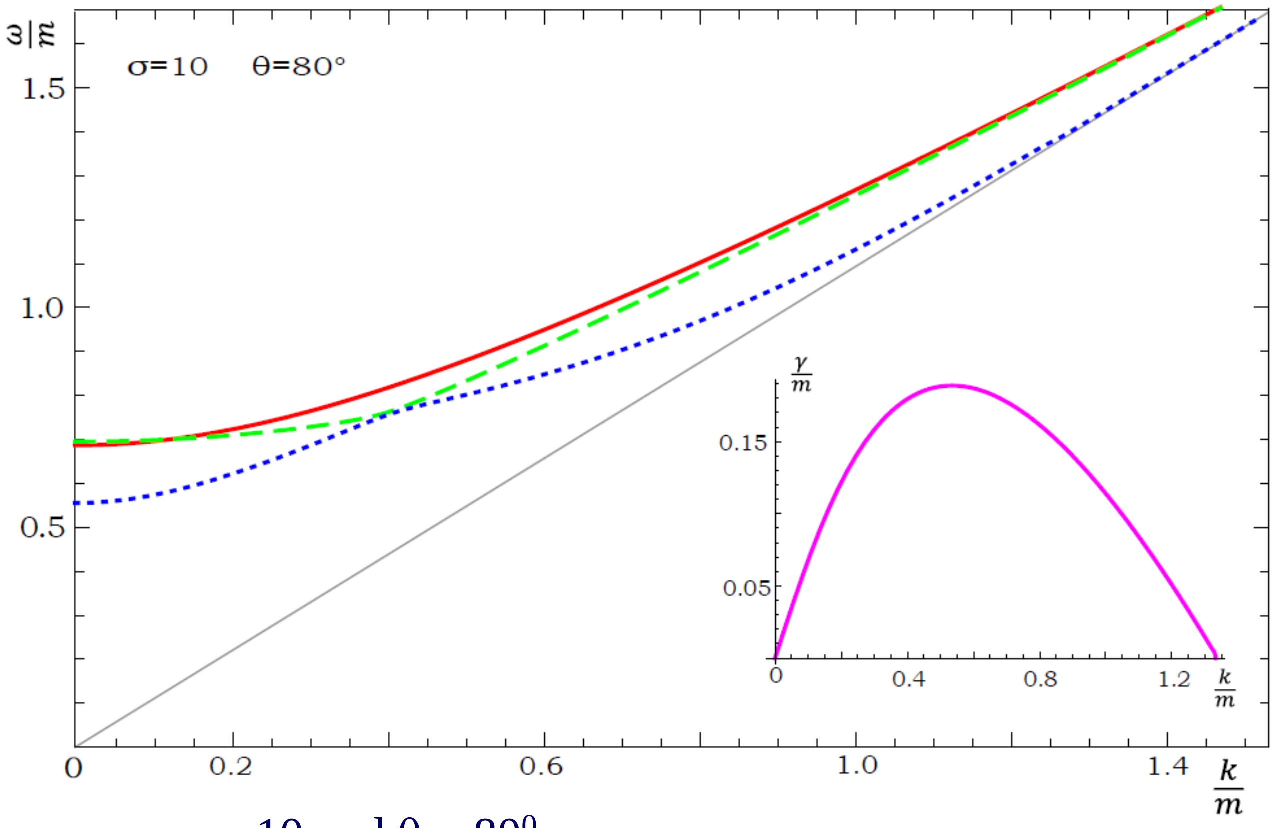
$$\gamma(\mathbf{k}) \underset{k>>\gamma}{\approx} \frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2} + 4\left(k_A^2 - k^2\right)} - \frac{\lambda}{k}\right)$$

$$\lambda \equiv \frac{\pi}{4}\left[1 - \frac{\xi}{2}\left(\frac{7}{3} - 5\cos^2\vartheta\right)\right]m^2$$

$$k_A \equiv m\Re\sqrt{\frac{\xi}{3}\left(2\cos^2\vartheta - 1\right)}$$

Finite prolateness

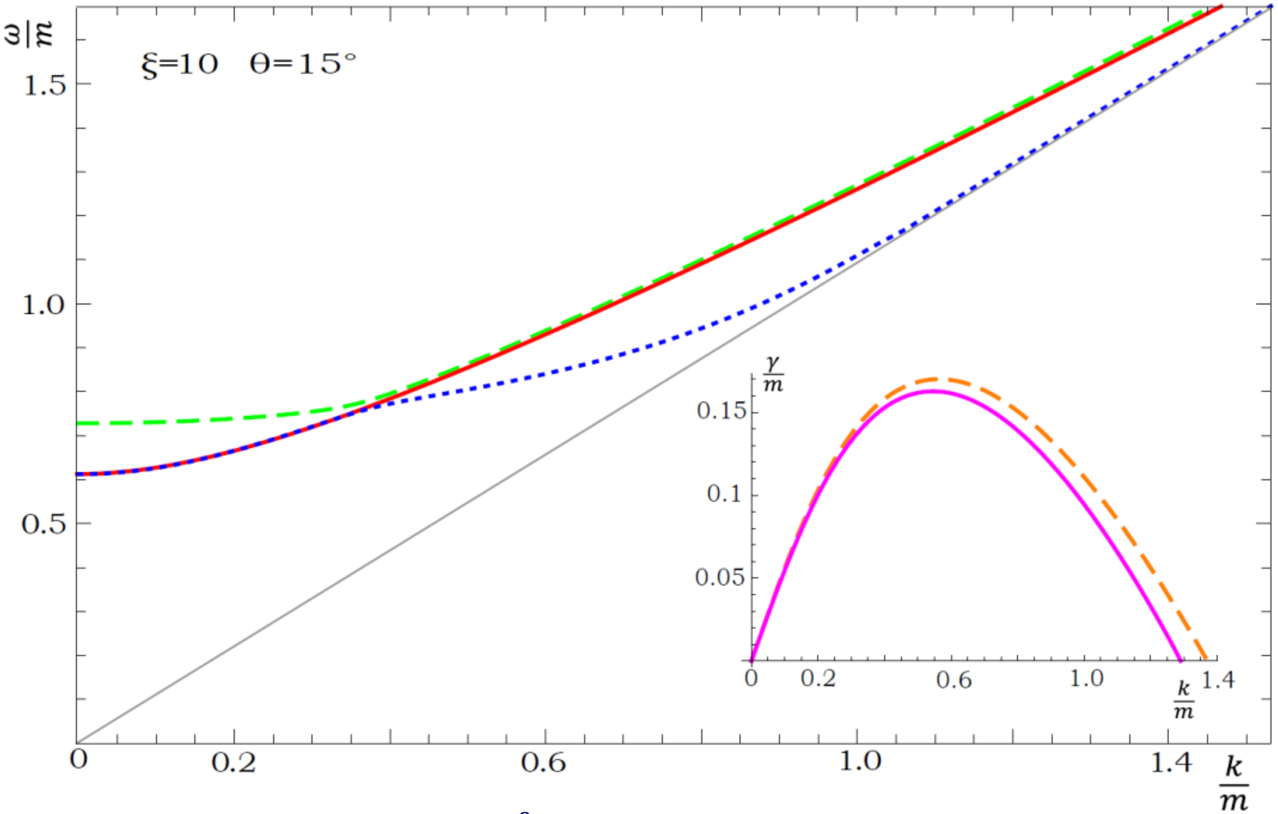
8 (numerical) solutions (6 real and 2 imaginary modes)



$\sigma = 10$ and $\theta = 80^0$

Finite oblateness

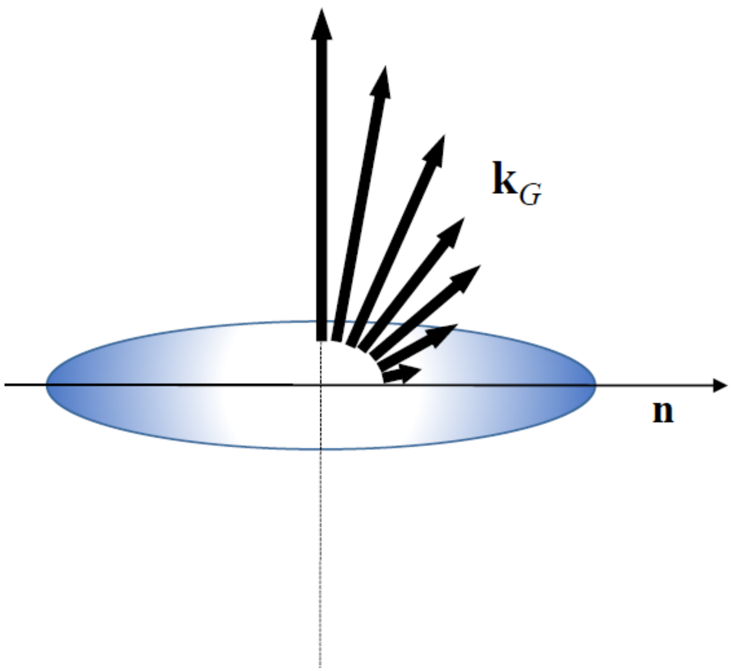
10 (numerical) solutions (6 real and 4 imaginary modes)



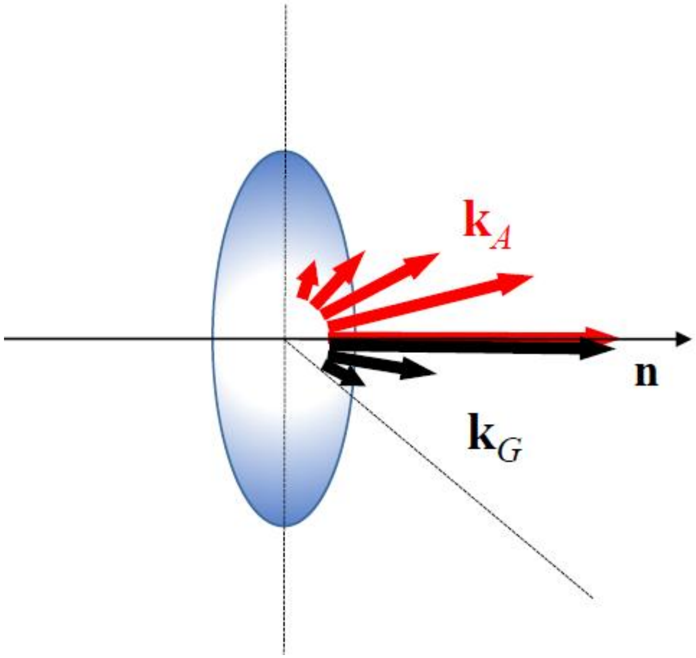
$\xi = 10$ and $\theta = 15^0$

Prolate versus oblate

Maximal wave vectors of unstable modes



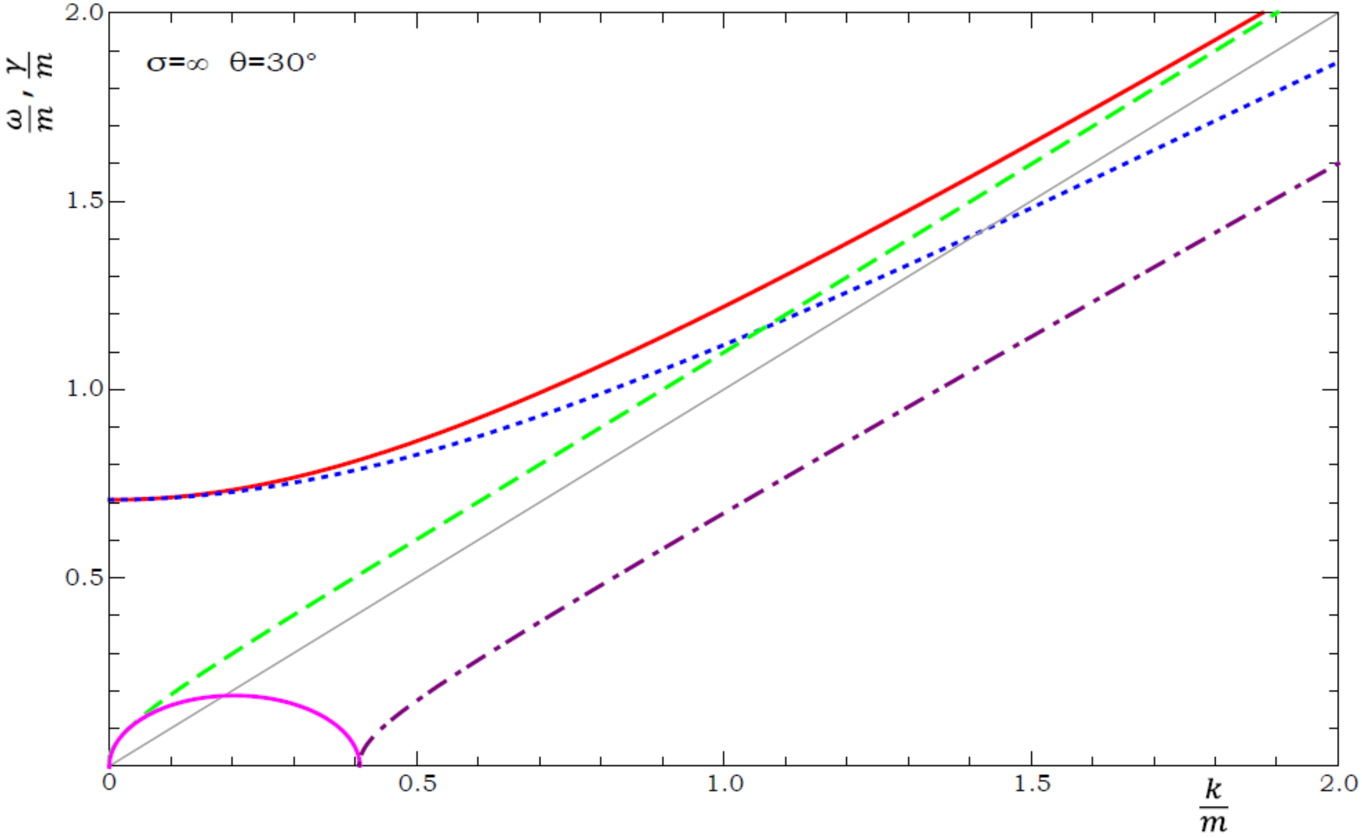
prolate



oblate

Extremely prolate QGP

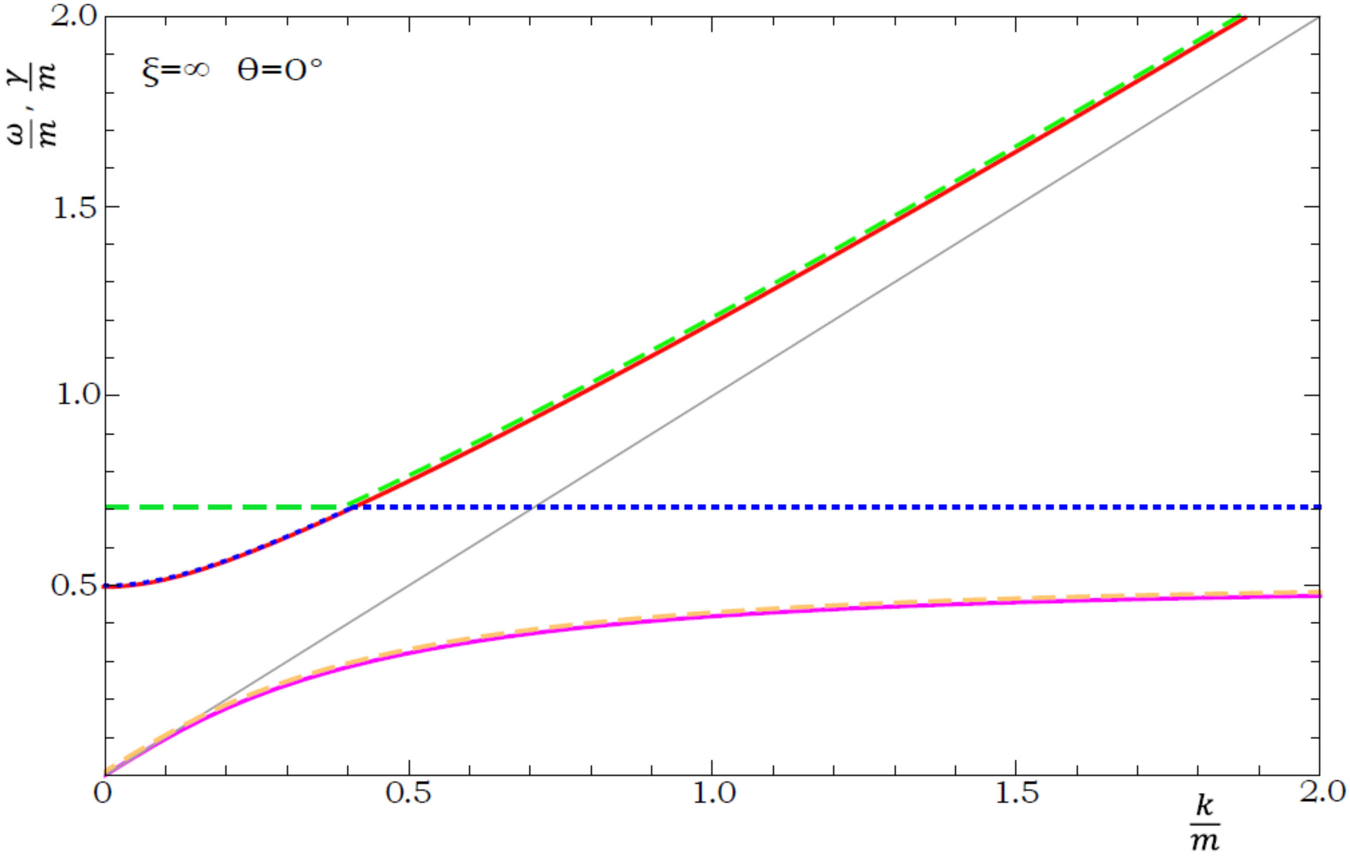
$f(\mathbf{p}) \sim \delta(p_T)$ 8 (analytic) solutions



$\theta = 30^0$

Extremely oblate QGP

$f(\mathbf{p}) \sim \delta(p_L)$ 8 or 10 solutions

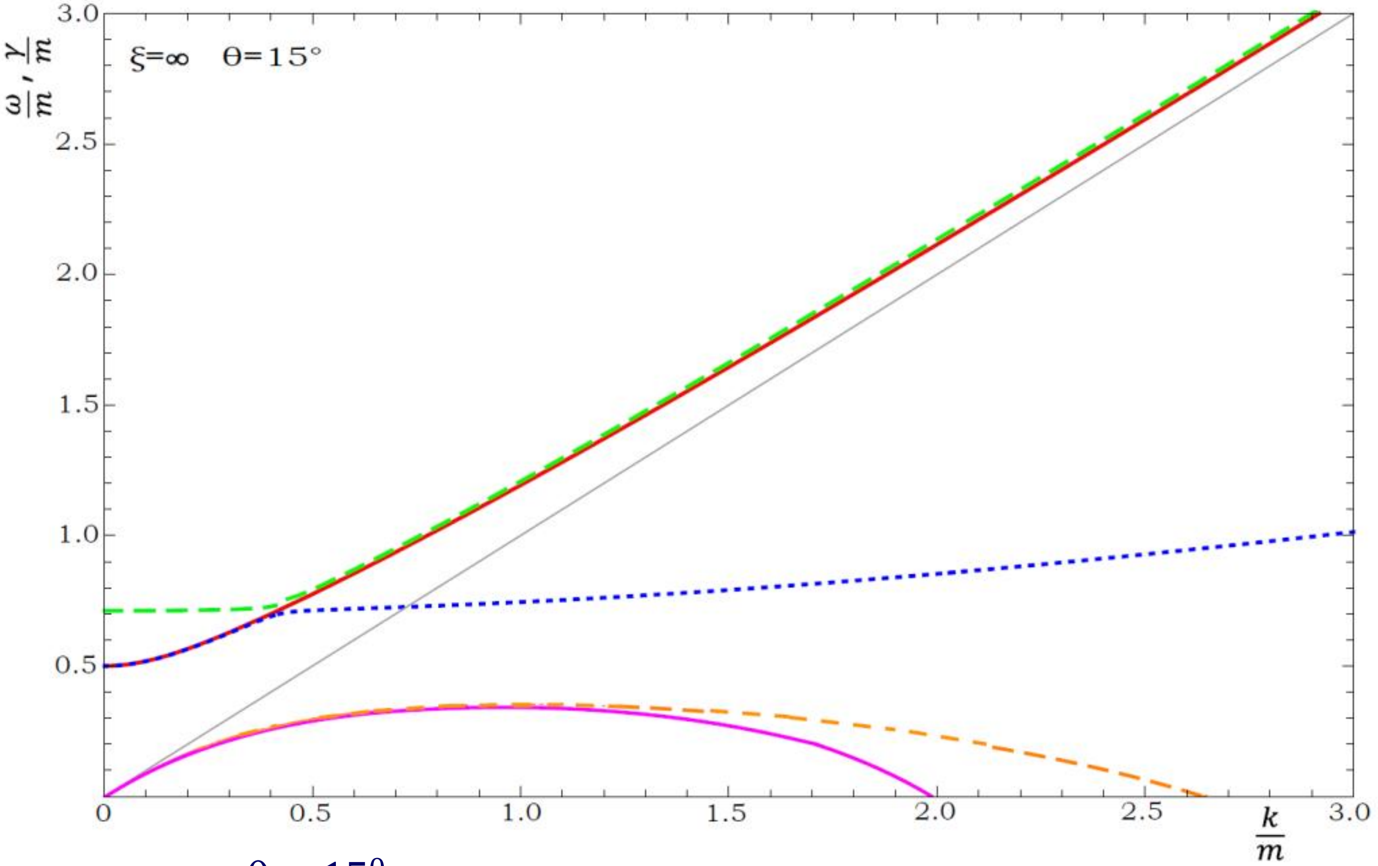


$\theta = 0^0$

How to connect the lines?

Extremely oblate QGP

$f(\mathbf{p}) \sim \delta(p_L)$ 8 or 10 solutions



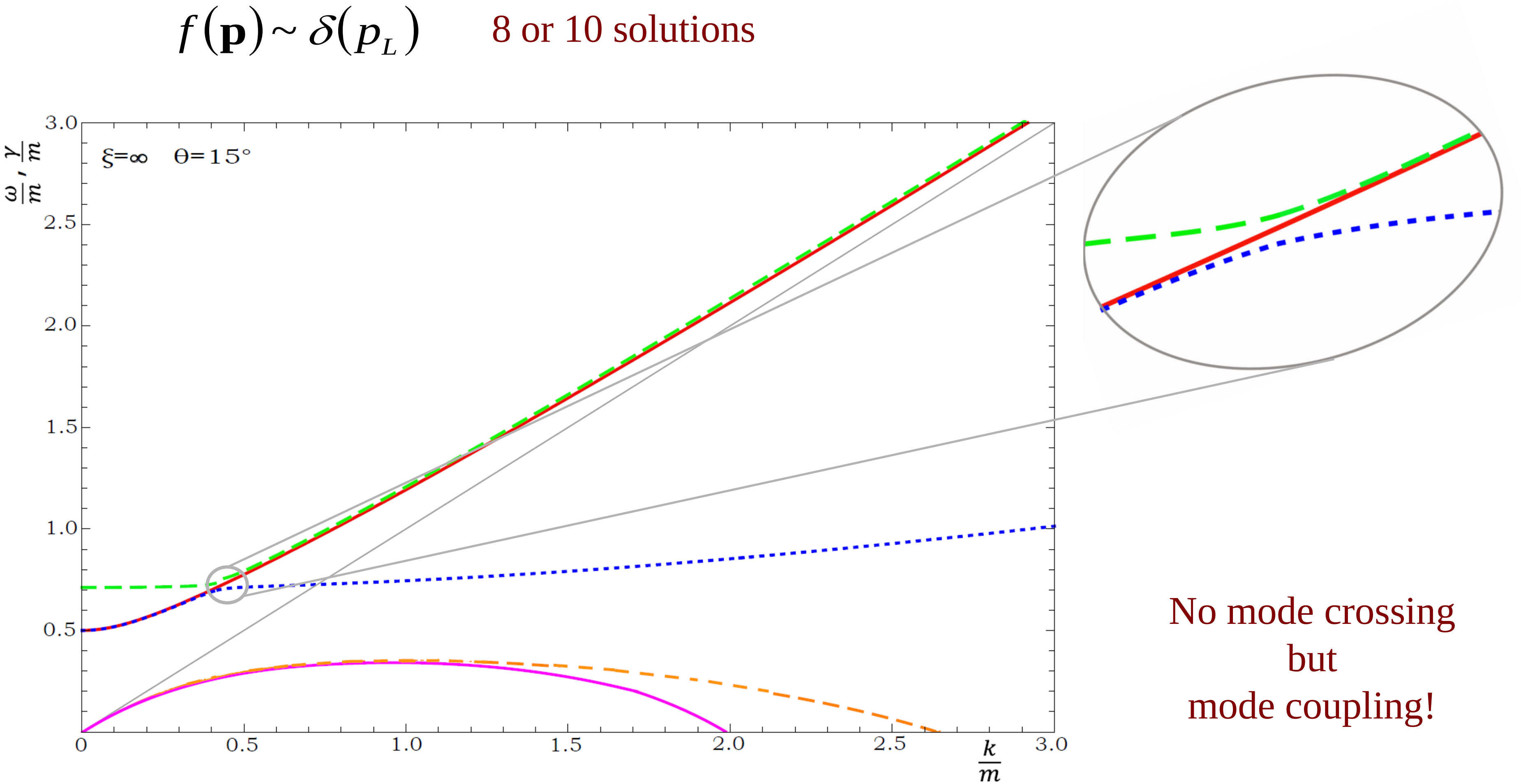
θ = 15^0

Mode crossing ?

Extremely oblate QGP

$$f(\mathbf{p}) \sim \delta(p_L)$$

8 or 10 solutions



No mode crossing
but
mode coupling!

$\theta = 15^0$

Number of solutions

The number of modes for each system

Momentum distribution	Number of real modes	Number of imaginary modes	Total number of modes	Maximal number of modes
extremely prolate	$6+2\Theta(k-k_{\mathrm{p}})$	$2\Theta(k_{\mathrm{p}}-k)$	8	8
weakly prolate	6	$2\Theta(k_{\mathrm{C}}-k)$	$6+2\Theta(k_{\mathrm{C}}-k)$	8
isotropic	6	0	6	6
weakly oblate	6	$2\Theta(k_{\mathrm{A}}-k)+2\Theta(k_{\mathrm{C}}-k)$	$6+2\Theta(k_{\mathrm{A}}-k)+2\Theta(k_{\mathrm{C}}-k)$	10
extremely oblate	6	$2\Theta(k_{\mathrm{oA}}-k)+2\Theta(k_{\mathrm{oG}}-k)$	$6+2\Theta(k_{\mathrm{oA}}-k)+2\Theta(k_{\mathrm{oG}}-k)$	10

Conclusions

- Systematic analysis of the complete mode spectrum is performed for any anisotropy.
- The number of modes is found in every case.
- Analytical and numerical solutions are given.
- There is no anisotropy threshold for existence of instabilities.
- Complete spectrum of modes is needed to compute various plasma characteristics e.g. the energy loss in anisotropic QGP.

for more see:

M. Carrington, K. Deja and St. Mrówczyński, arXiv:1407.2764, to appear in Phys. Rev. **C**