Chromohydrodynamics of the Quark-Gluon Plasma

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C. Manuel & St. M., Phys. Rev. **D74** (2006) 105003

Motivation

Fluid equations are frequently applied to the electromagnetic plasma

Why not to the quark-gluon plasma?

Long but quiet history of chromohydrodynamics

- Kajantie & Montonen 1980
- Holm & Kupershmidt 1984
- Mrówczyński 1988
- Bhatt, Kaw & Parikh 1989
- Jackiw, Nair & Pi 2000
- Manuel & Mrówczyński 2003

Real hydrodynamics of QGP

Real hydrodynamics describes systems in local thermodynamic equilibrium

- Local equilibrium is a state of maximal local entropy $C = 0$
- Local equilibrium is achieved at the time scale of hard collisions $\sim \left(g^4 \ln(1/g)\, T\right)^{-1}$
- Color redistribution occurs at the time scale of hard collisions $\sim \left(g^2 \ln(1/g)\, T\right)^{-1}$
- Whitening occurs at the time scale of soft collisions $\sim \ln(1/g)/T$

Manuel & Mrówczyński, *Local Equilibrium of the Quark-Gluon Plasma*, Phys. Rev. **D68** (2003) 094010	Manuel & Mrówczyński, *Whitening of the Quark-Gluon Plasma*, Phys. Rev. **D70** (2004) 094019

Real hydrodynamics of QGP cont.

Real hydrodynamics of QGP is colorless

$$t_{\mathrm{hydro}} \geq t_{\mathrm{hard\ coll}} \sim \left(g^4 \ln(1/g)\, T\right)^{-1}$$

There is no QGP analog of magnetohydrodynamics of EM plasma

Delayed mutual equilibration of electrons and ions

$$m_{\mathrm{ion}} >> m_{\mathrm{electron}}$$

Chromohydrodynamics

Fluid approach to QGP at time scale shorter than soft collisions

$$t < t_{\text{soft coll}} \sim \left(g^2 \ln(1/g)\, T\right)^{-1}$$

Collisionless regime of kinetic theory

Kinetic theory

Distribution functions of quarks $Q(p,x)$ and antiquarks $\overline{Q}(p,x)$ are $N_c \times N_c$ matrices

Distribution function of gluons $G(p,x)$ is $(N_c^2-1)\times(N_c^2-1)$ matrix

Transport equations

fundamental:

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q\} = C \quad \text{quarks}$$

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu \overline{Q}\} = \overline{C} \quad \text{antiquarks}$$

adjoint:

$$p_\mu D^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu\nu}, \partial_p^\nu G\} = C_g \quad \text{gluons}$$

$D^\mu \equiv \partial^\mu - ig[A^\mu, ...]$

free streaming — mean-field force — collisions

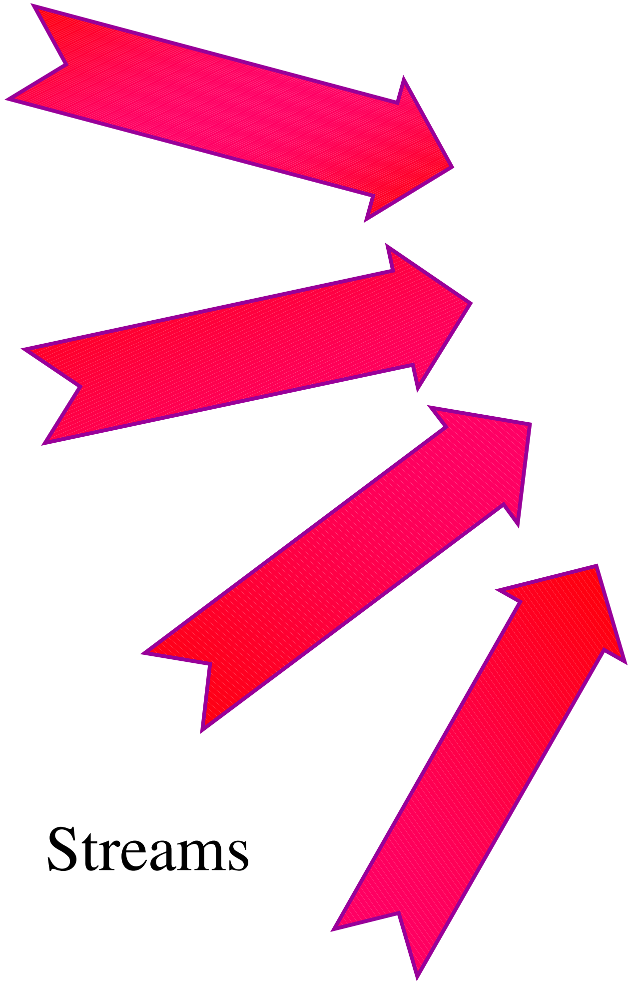
$D^\mu \equiv \partial^\mu F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$

Mean-field generation

$$D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]$$

Collisionless limit: $C = \overline{C} = C_g = 0$

Streams



Streams

$$Q(p,x) = \sum_{\alpha} Q_{\alpha}(p,x)$$

$$p_{\mu} D^{\mu} Q_{\alpha} - \frac{g}{2} p^{\mu} \{ F_{\mu\nu}, \partial_{p}^{\nu} Q_{\alpha} \} = 0$$

Streams ‘talk’ to each other only via mean field.

$$D_{\mu} F^{\mu\nu} = \sum_{\alpha} j_{\alpha}^{\nu}$$

Derivation of chromohydrodynamic equations

Transport equation of quark distribution function of the α-th stream $Q_\alpha(p,x)$

$$p_\mu D^\mu Q_\alpha(p,x) - \frac{g}{2} p^\mu \{ F_{\mu\nu}, \partial_p^\nu Q_\alpha(p,x) \} = 0$$

Taking into account antiquarks and gluons is straightforward

$$\int dP \quad \boxed{}$$

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0)\,\delta(p^2)$$

$$D_\mu n_\alpha^\mu(x) = 0$$

$$n_\alpha^\mu(x) \equiv \int dP\, p^\mu Q_\alpha(p,x)$$

Derivation of chromohydrodynamic equations cont.

$$p_\mu D^\mu Q_\alpha(p,x) - \frac{g}{2} p^\mu \{ F_{\mu\nu}, \partial_p^\nu Q_\alpha(p,x) \} = 0$$

$$\int dP\, p^\mu$$

$$D_\mu T_\alpha^{\mu\nu}(x) - \frac{g}{2} \{ F_{\mu\nu}, n_\alpha^\mu(x) \} = 0$$

$$T_\alpha^{\mu\nu}(x) \equiv \int dP\, p^\mu p^\nu Q_\alpha(p,x)$$

$$m = 0 \;\Rightarrow\; T^\mu_{\alpha\,\mu} = 0$$

Chromohydrodynamic equations

$$D_\mu n_\alpha^\mu(x) = 0$$

$$D_\mu T_\alpha^{\mu\nu}(x) - \frac{g}{2}\{F^{\mu\nu}, n_{\alpha\,\mu}(x)\} = 0$$

Postulated form of $n_\alpha^\mu(x)$ and $T_\alpha^{\mu\nu}(x)$:

isotropy in the local rest frame

$$n_\alpha^\mu(x) = n_\alpha(x)\, u_\alpha^\mu(x)$$

$$T_\alpha^{\mu\nu}(x) = \frac{1}{2}(\varepsilon_\alpha(x) + p_\alpha(x))\{u_\alpha^\mu(x), u_\alpha^\nu(x)\} - p_\alpha(x)\, g^{\mu\nu}$$

$n_\alpha(x)$, $\varepsilon_\alpha(x)$, $p_\alpha(x)$, $u_\alpha^\mu(x)$ matrices!

Chromohydrodynamic equations cont.

Gauge transformations of $n_\alpha(x), \varepsilon_\alpha(x), p_\alpha(x), u_\alpha^\mu(x)$

$$n_\alpha(x) \rightarrow U(x) n_\alpha(x) U^{-1}(x)$$

$n_\alpha(x), \varepsilon_\alpha(x), p_\alpha(x), u_\alpha^\mu(x)$ cannot be simultaneously diagonalized

$$n_\alpha^\mu(x) \equiv \int dP\, p^\mu Q_\alpha(p,x)$$

$$Q_\alpha(p,x) \rightarrow U(x) Q_\alpha(p,x) U^{-1}(x)$$

Chromohydrodynamic equations cont.

index α suppressed

$$D_\mu n^\mu(x) = 0$$

1 matrix equation

$$D_\mu T^{\mu\nu}(x) - \frac{g}{2}\{F^{\mu\nu}, n_\mu(x)\} = 0$$

4 matrix equations

5 matrix equations

$$n^\mu(x) = n(x)\,u^\mu(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2}(\varepsilon(x) + p(x))\{u^\mu(x), u^\mu(x)\} - p(x)\,g^{\mu\nu}$$

$n(x), \varepsilon(x), p(x), u^\mu(x)$ 6 matrix functions

$$u^\mu(x)\,u_\mu(x) = 1$$

To close the system of equations: $\nabla p = 0$ or $\varepsilon = 3p \Leftarrow T^\mu_\mu = 0$

Linear response approximation

Small perturbation of the space-time homogeneous & colorless state

$$n(x) = \tilde{n} + \delta n(x), \quad \varepsilon(x) = \tilde{\varepsilon} + \delta\varepsilon(x),$$

$$p(x) = \tilde{p} + \delta p(x), \quad u^{\mu}(x) = \tilde{u}^{\mu} + \delta u^{\mu}(x)$$

$\tilde{n}, \tilde{\varepsilon}, \tilde{p}, \tilde{u}^{\mu}$ unit matrices in color space

$$\tilde{n} >> \delta n, \quad \tilde{\varepsilon} >> \delta\varepsilon, \quad \tilde{p} >> \delta p, \quad \tilde{u}^{\mu} >> \delta u^{\mu}$$

$$F^{\mu\nu} \sim A^{\mu} \sim \delta n$$

index α suppressed

Linearized chromohydrodyanmic equations

Continuity equation

$$(D_\mu \delta n)\,\tilde{u}^\mu + \tilde{n}\, D_\mu \delta u^\mu = 0$$

Euler equation

$$(\tilde{\varepsilon} + \tilde{p})\tilde{u}^\mu D_\mu \delta u^\nu - (D^\nu - \tilde{u}^\nu \tilde{u}^\mu D_\mu)\delta p - g\tilde{n}\,\tilde{u}_\mu F^{\mu\nu} = 0$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - \tilde{u}^\mu \tilde{u}^\nu$$

$\delta\varepsilon$ exactly eliminated

$D^\nu \delta p \approx 0$ dynamics dominated by the mean field

index α suppressed

Solutions of the linearized equations

- $D^{\mu} \rightarrow \partial^{\mu}$ full linearization $A^{\mu} \sim \delta n$
- Fourier transformations

continuity	$k_{\mu}\tilde{u}^{\mu}\,\delta n(k) + \tilde{n}\,k_{\mu}\delta u^{\mu}(k) = 0$
Euler	$i(\tilde{\varepsilon} + \tilde{p})\,\tilde{u}^{\mu}k_{\mu}\delta u^{\nu}(k) - g\tilde{n}\,\tilde{u}_{\mu}F^{\mu\nu}(k) = 0$

$\partial^{\nu}\delta p \approx 0$

Solutions

$$\delta n(k) = ig\,\frac{\tilde{n}^{2}}{\tilde{\varepsilon} + \tilde{p}}\,\frac{\tilde{u}_{\nu}\,k_{\mu}}{(\tilde{u}\cdot k)^{2}}\,F^{\mu\nu}(k)$$

$$\delta u^{\mu}(k) = ig\,\frac{\tilde{n}}{\tilde{\varepsilon} + \tilde{p}}\,\frac{\tilde{u}_{\nu}}{\tilde{u}\cdot k}\,F^{\mu\nu}(k)$$

index α suppressed

Color current

$$j^{\mu}(x) = -\frac{g}{2}\left(n(x)u^{\mu}(x) - \frac{1}{N_c}\mathrm{Tr}[\,n(x)u^{\mu}(x)] \right)$$

$$j^{\mu}(x) = \tilde{j}^{\mu} + \delta j^{\mu}(x), \qquad \tilde{j}^{\mu} = 0$$

$$\delta j^{\mu}(x) = -\frac{g}{2}\left(\tilde{n}\,\delta u^{\mu}(x) + \tilde{u}^{\mu}\delta n(x) \right)$$

$\mathrm{Tr}[F^{\mu\nu}] = 0$

polarization tensor $\Pi^{\mu\nu}(x, y) = \dfrac{\delta j^{\mu}(x)}{\delta A_{\nu}(y)}$

index α suppressed

Polarization tensor

index α restored

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\varepsilon_\alpha + \tilde{p}_\alpha} \frac{(\tilde{u}_\alpha \cdot k)(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - k^2 \tilde{u}_\alpha^\mu \tilde{u}_\alpha^\nu - (\tilde{u}_\alpha \cdot k)^2 g^{\mu\nu}}{(\tilde{u}_\alpha \cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k)\,, \qquad k_\mu \Pi^{\mu\nu}(k) = 0$$

Hydrodynamics vs. kinetic theory

Kinetic theory result

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int dP f(p) \frac{(p\cdot k)(p^\mu k^\nu + p^\nu k^\mu) - k^2 p^\mu p^\nu - (p\cdot k)^2 g^{\mu\nu}}{(p\cdot k)^2}$$

$$f(p) = \sum_\alpha \tilde{n}_\alpha \tilde{u}^0_\alpha \delta^{(3)}(\mathbf{p} - m_\alpha \tilde{\mathbf{u}}_\alpha)$$

$$m_\alpha \equiv \frac{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha}{\tilde{n}_\alpha}$$

Hydrodynamic result

Effect of pressure gradients

The set of fluid equations is closed by the relation $p_\alpha(x) = \frac{1}{3}\varepsilon_\alpha(x)$

$$\Pi^{\mu\nu}(k) = -\frac{3g^2}{8}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha}\Bigg[\frac{(\tilde{u}_\alpha\cdot k)(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - k^2\tilde{u}_\alpha^\mu\tilde{u}_\alpha^\nu - (\tilde{u}_\alpha\cdot k)^2 g^{\mu\nu}}{(\tilde{u}_\alpha\cdot k)^2}$$

$$-\frac{(\tilde{u}_\alpha\cdot k)k^2(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - (\tilde{u}_\alpha\cdot k)^2 k^\mu k^\nu - k^4\tilde{u}_\alpha^\mu\tilde{u}_\alpha^\nu}{k^2 + 2(\tilde{u}_\alpha\cdot k)^2}\Bigg]$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k)\,, \qquad k_\mu\Pi^{\mu\nu}(k) = 0$$

Application

Collective modes in the two-stream systems

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$ chromodielectric tensor

$$\det[\mathbf{k}^2\delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\omega_p^2 \equiv \frac{g^2}{2}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha}$$

$$\varepsilon^{ij}(\omega,\mathbf{k}) = \delta^{ij}\left(1 - \frac{\omega_p^2}{\omega^2}\right) - \frac{g^2}{2\omega^2}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha}\left(\frac{\tilde{v}_\alpha^i k^j + \tilde{v}_\alpha^j k^i}{\omega - \mathbf{k}\cdot\tilde{\mathbf{v}}_\alpha} + \frac{(\omega^2 - \mathbf{k}^2)\tilde{v}_\alpha^i \tilde{v}_\alpha^j}{(\omega - \mathbf{k}\cdot\tilde{\mathbf{v}}_\alpha)^2}\right)$$

Two-stream system

$$\tilde{\mathbf{v}}_+ , \tilde{n}_+ , \tilde{\varepsilon}_+ , \tilde{p}_+ \qquad \tilde{\mathbf{v}}_- , \tilde{n}_- , \tilde{\varepsilon}_- , \tilde{p}_-$$

$$\tilde{\mathbf{v}} \equiv \tilde{\mathbf{v}}_+ = -\tilde{\mathbf{v}}_- \qquad \tilde{\mathbf{u}} = \tilde{\gamma}\, \tilde{\mathbf{v}}$$

$$\tilde{n} \equiv \tilde{n}_+ = \tilde{n}_- , \quad \tilde{\varepsilon} \equiv \tilde{\varepsilon}_+ = \tilde{\varepsilon}_- , \quad \tilde{p} \equiv \tilde{p}_+ = \tilde{p}_-$$

$$\varepsilon^{ij}(\omega,\mathbf{k}) = \delta^{ij}\Big(1 - \frac{\omega_p^2}{\omega^2}\Big) - \frac{g^2}{2\omega^2}\frac{\tilde{n}^2}{\tilde{\varepsilon}+\tilde{p}}\Big(\frac{\tilde{v}^i k^j + \tilde{v}^j k^i}{\omega - \mathbf{k}\cdot\mathbf{v}} + \frac{(\omega^2 - \mathbf{k}^2)\tilde{v}^i \tilde{v}^j}{(\omega - \mathbf{k}\cdot\mathbf{v})^2}$$
$$- \frac{\tilde{v}^i k^j + \tilde{v}^j k^i}{\omega + \mathbf{k}\cdot\mathbf{v}} + \frac{(\omega^2 - \mathbf{k}^2)\tilde{v}^i \tilde{v}^j}{(\omega + \mathbf{k}\cdot\mathbf{v})^2}\Big)$$

$\mathbf{k} \perp \mathbf{v}$

$$\tilde{\mathbf{v}} = (0, 0, \tilde{v}), \qquad \mathbf{k} = (k, 0, 0)$$

Dispersion equation

$$(\omega^2 - \omega_p^2)(\omega^2 - \omega_p^2 - k^2)\left(\omega^2 - \omega_p^2 - k^2 - \lambda^2 \frac{k^2 - \omega^2}{\omega^2}\right) = 0$$

$$\lambda \equiv \omega_p \tilde{v}$$

stable longitudinal mode

$$\omega^2 = \omega_p^2$$

stable transverse modes

$$\omega^2 = \omega_p^2 + k^2$$

$$\omega_+^2 = \frac{1}{2}\left(\omega_p^2 - \lambda^2 + k^2 + \sqrt{(\omega_p^2 - \lambda^2 + k^2)^2 + 4\lambda^2 k^2}\right)$$

unstable transverse, filamentation mode

$$\omega_-^2 = \frac{1}{2}\left(\omega_p^2 - \lambda^2 + k^2 - \sqrt{(\omega_p^2 - \lambda^2 + k^2)^2 + 4\lambda^2 k^2}\right)$$

k || v

$$\tilde{\mathbf{v}} = (0,0,\tilde{v}), \qquad \mathbf{k} = (0,0,k)$$

Dispersion equation

$$(\omega^2 - \omega_p^2 - k^2)^2 \left(1 - \frac{\omega_0^2}{(\omega - k\tilde{v})^2} + \frac{\omega_0^2}{(\omega + k\tilde{v})^2} \right) = 0$$

$$\omega_0^2 \equiv \omega_p^2 (1 - \tilde{v}^2)/2$$

stable transverse modes

$$\omega^2 = \omega_p^2 + k^2$$

stable longitudinal mode

$$\omega_+^2 = \omega_0^2 + k^2\tilde{v}^2 + \sqrt{\omega_0^2 + 4k^2\tilde{v}^2}$$

unstable longitudinal mode

$$\omega_+^2 = \omega_0^2 + k^2\tilde{v}^2 - \sqrt{\omega_0^2 + 4k^2\tilde{v}^2}$$

Conclusions

Chromohydrodynamics is much simpler than the kinetic theory.

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Chromohydrodynamics is dynamically nontrivial.

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Chromohydrodynamics provides alternative approach
to study nonequilibrium QGP.