Instabilities Driven Equilibration of the Quark-Gluon Plasma
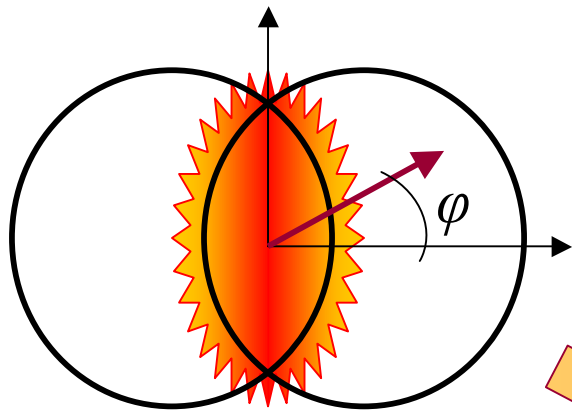
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& Institute for Nuclear Studies, Warsaw, Poland

- Review focused on recent developments -
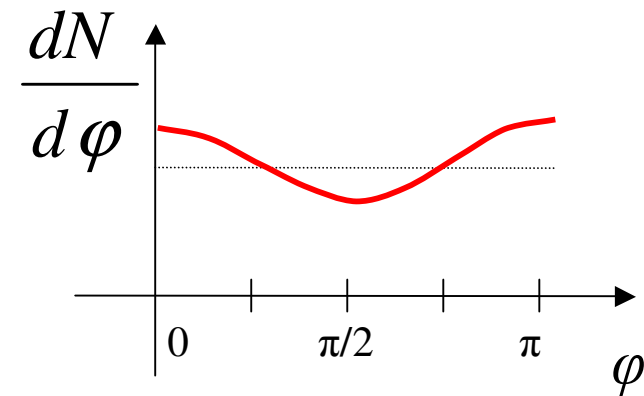
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Evidence of the early stage equilibration

Success of hydrodynamic models in describing elliptic flow

Hydrodynamics

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\mathbf{v} = -\frac{\nabla p}{\rho}$$

Hydrodynamic requires local thermodynamical equilibrium!

Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

Eccentricity decays due to the free streaming!

$$\varepsilon \searrow \;\Rightarrow\; v_2 \searrow$$

$$t_{eq} \leq 0.6 \;\; fm/c$$

time of equilibration

U. Heinz, AIP Conf. Proc.**739,** 163 (2004)

Collisions are too slow

Time scale of hard parton-parton scattering

hard scattering ~ momentum transfer of order of T

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g) T}$$

either single hard scattering or multiple soft scatterings

$$t_{\text{eq}} \approx t_{\text{hard}} \geq 2.6 \ \text{fm}/c$$

R. Baier, A.H. Mueller, D. Schiff & D.T. Son, Phys. Lett. **B539**, 46 (2002)

Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

fluctuation

Instability

$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

stable configuration

A_0

$A(t)$

unstable configuration

A_0

$A(t)$

Terminology

Plasma instabilities – interplay of particles and classical fields

Quantum Field Theory – no particles, no classical fields

$$p_{\text{hard}} \sim T$$

- particles – hard excitations, hard modes

- classical fields – highly populated soft excitations, soft modes

$$\sim 1/g^2 \qquad p_{\text{soft}} \sim gT$$

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Plasma instabilities

▶ **instabilities in configuration space – hydrodynamic instabilities**

▶ **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

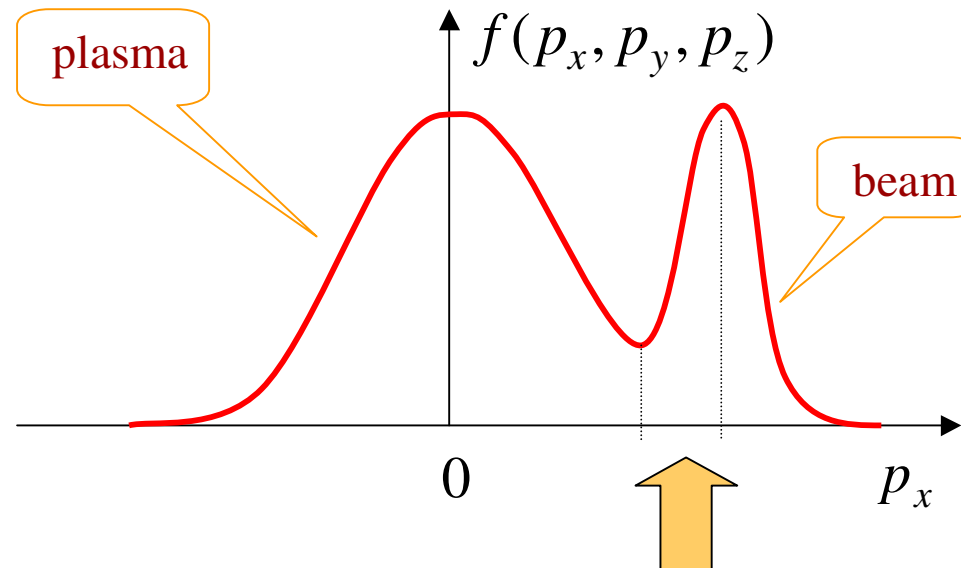$$f(\mathbf{p}) \;\; \text{is} \;\; \text{not} \;\; \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

▶ **longitudinal modes** $-$ $\mathbf{k} \parallel \mathbf{E}, \ \delta\rho \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

▶ **transverse modes** $-$ $\mathbf{k} \perp \mathbf{E}, \ \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

$\mathbf{E}$ – electric field, $\mathbf{k}$ – wave vector, ρ – charge density, $\mathbf{j}$ - current
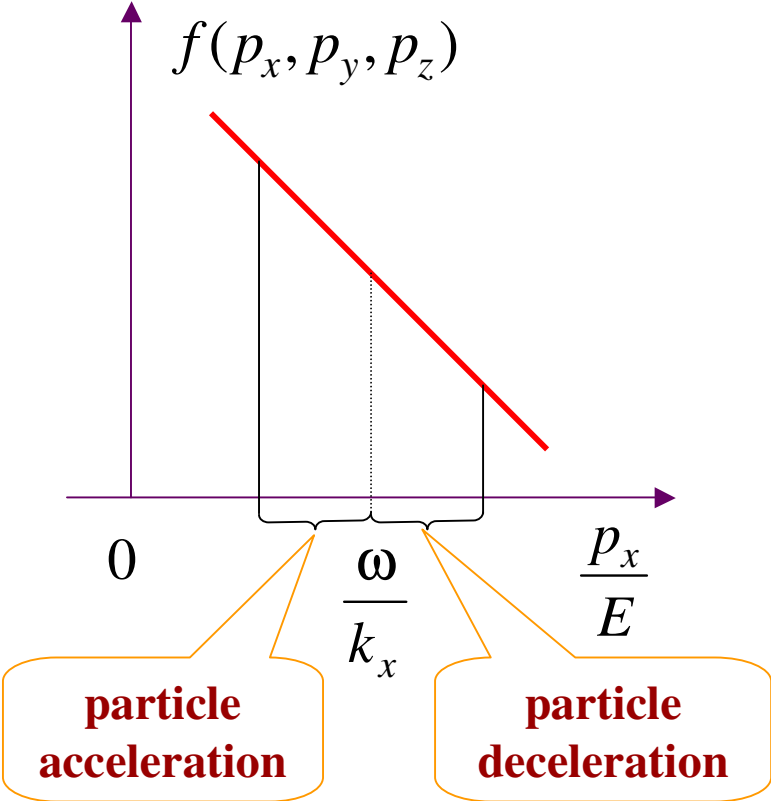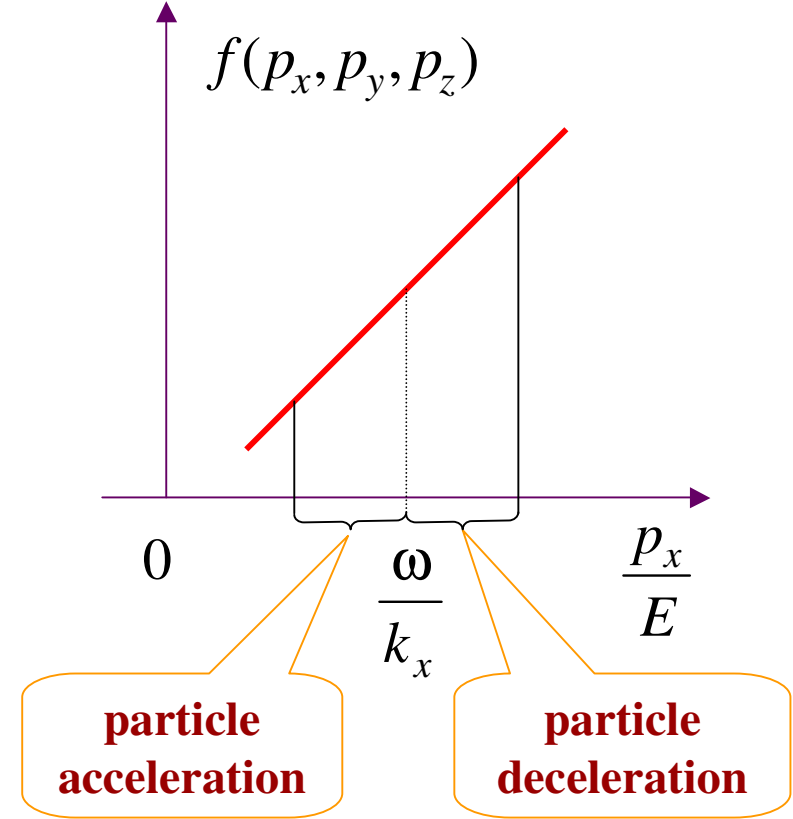
Logitudinal modes

unstable configuration

Energy is transferred from particles to fields

Logitudinal modes

Electric field decays - damping

Electric field grows - instability

$f(p_x, p_y, p_z)$

$\dfrac{\omega}{k_x}$

$\dfrac{p_x}{E}$

particle acceleration

particle deceleration

$f(p_x, p_y, p_z)$

$\dfrac{\omega}{k_x}$

$\dfrac{p_x}{E}$

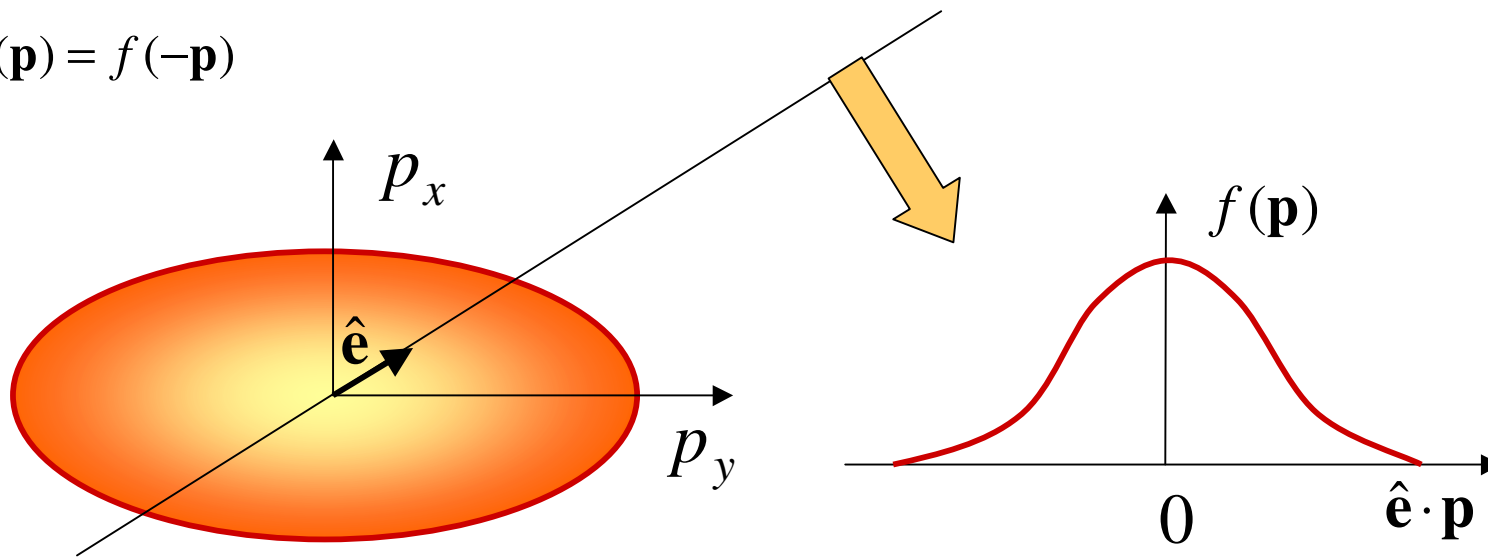**particle acceleration**

particle deceleration

$\dfrac{\omega}{k_x}$ - phase velocity of the electric field wave,

$\dfrac{p_x}{E}$ - particle's velocity

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Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution
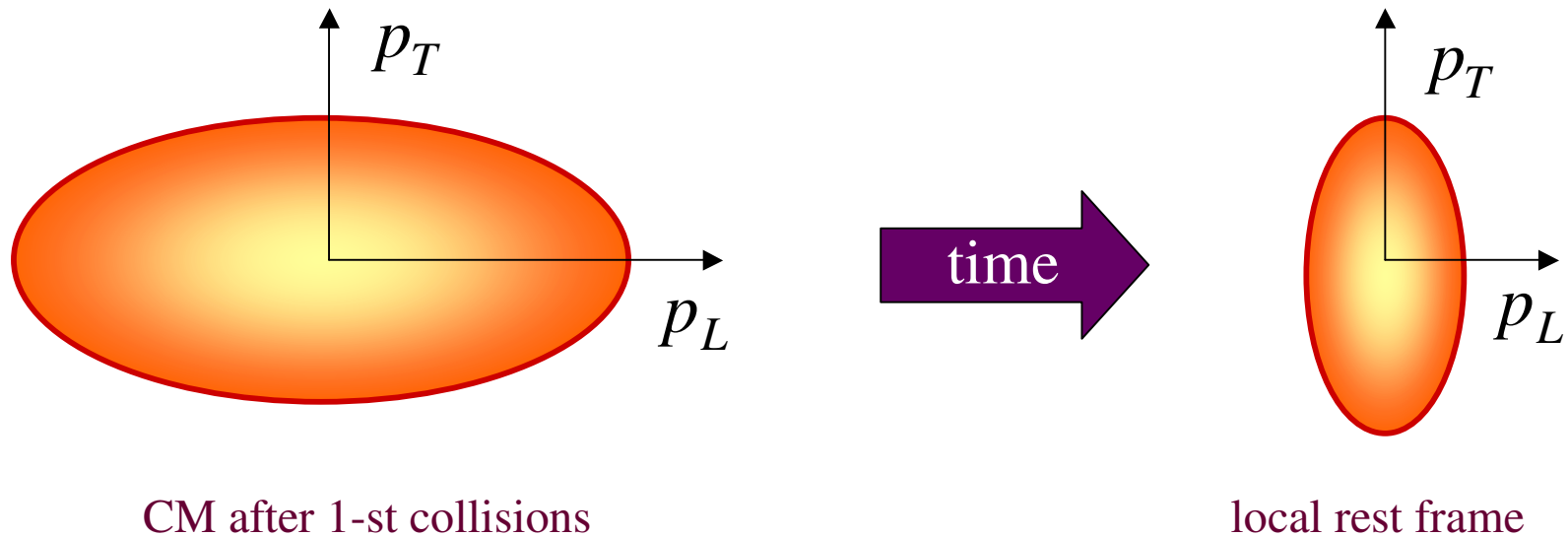
$$f(\mathbf{p}) = f(-\mathbf{p})$$

Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

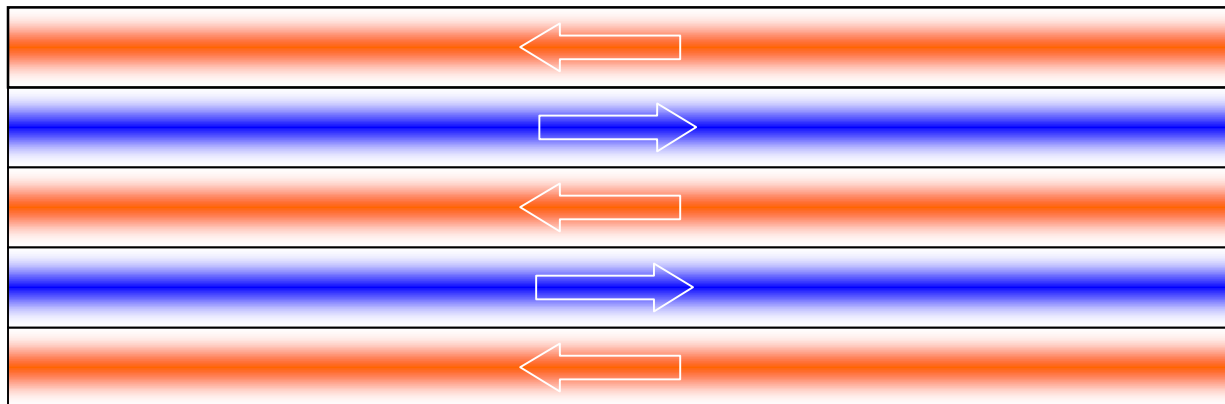**Parton momentum distribution is initially strongly anisotropic**

CM after 1-st collisions

time

local rest frame

$\left\langle j_a^\mu(x) \right\rangle = 0$ **but current fluctuations are finite**

$$\left\langle j_a^\mu(x_1)\, j_b^\nu(x_2) \right\rangle = \frac{1}{2}\delta^{ab} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0$$
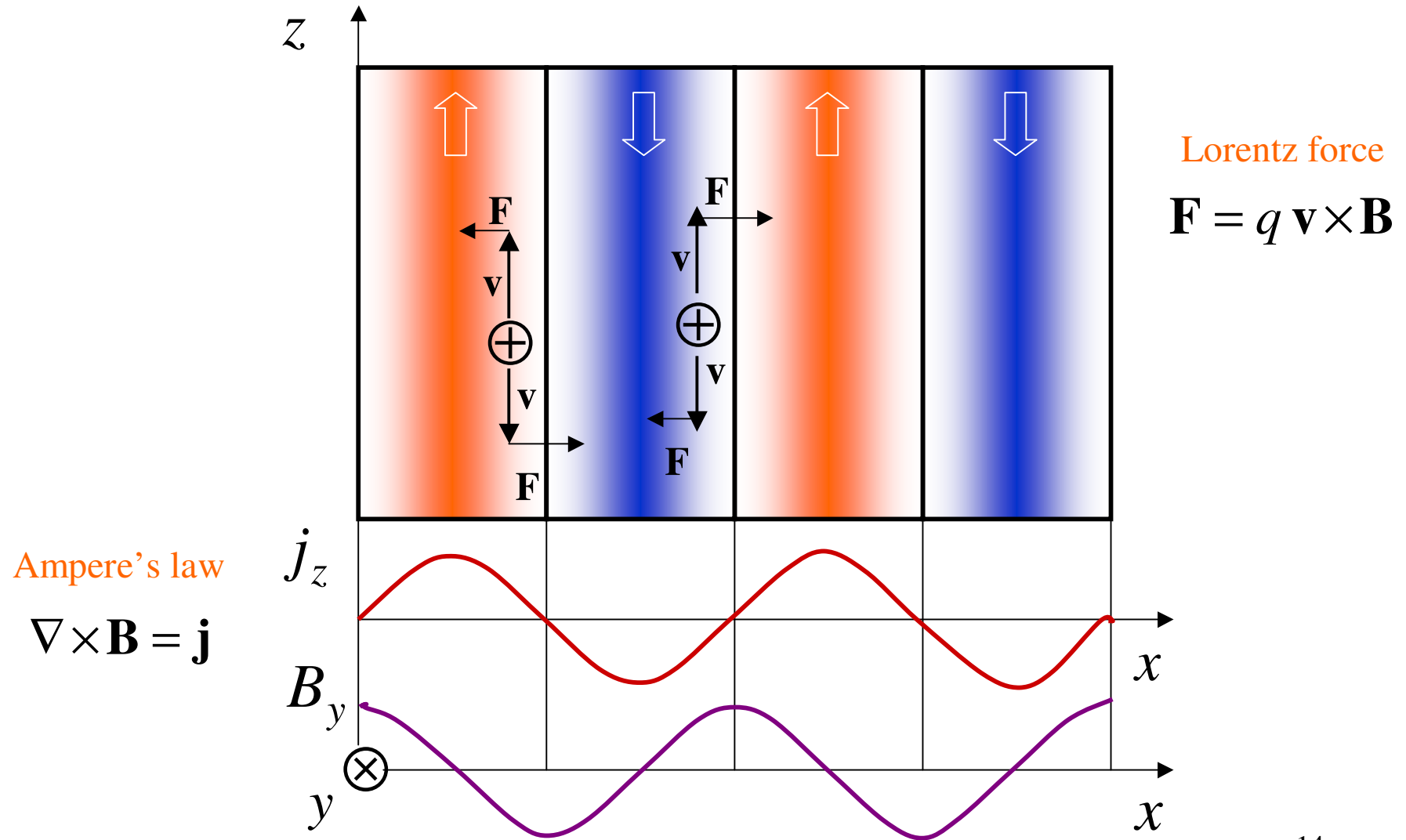
$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$

Direction of the momentum surplus

Mechanism of filamentation

Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

Ampere's law

$$\nabla \times \mathbf{B} = \mathbf{j}$$

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with $\mathrm{Im}\,\omega > 0$ $\Rightarrow$ $A^\mu(x) \sim e^{\mathrm{Im}\,\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. How to get it?

Transport theory – distribution functions

Distribution functions of quarks $Q(p,x)$ and antiquarks $\overline{Q}(p,x)$

are gauge dependent $N_c \times N_c$ matrices

The gauge transformation:

$$Q(p,x) \to U(x)Q(p,x)U^{-1}(x)$$

Distribution function of gluons $G(p,x)$ is $(N_c^2-1) \times (N_c^2-1)$ matrix

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu v}(x), \partial^v_p Q\} = C \qquad \text{quarks}$$

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu v}(x), \partial^v_p \overline{Q}\} = \overline{C} \qquad \text{antiquarks}$$

adjoint

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu v},(x)\partial^v_p G\} = C_g \qquad \text{gluons}$$

free streaming

mean-field force

collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu v} \equiv \partial^\mu A^v - \partial^v A^\mu - ig[A^\mu, A^v]$$

$$D_\mu F^{\mu v} = j^v[Q, \overline{Q}, G]$$

mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

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Transport theory - linearization

fluctuation

$$Q(p, x) = Q_0(p) + \delta Q(p, x)$$

stationary colorless state $\quad Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| \gg |\delta Q(p, x)|, \quad |\partial_p^\mu Q_0(p)| \gg |\partial_p^\mu \delta Q(p, x)|$$

Linearized transport equations

$$p_\mu D^\mu \delta Q(p, x) - g p^\mu F_{\mu v}(x) \partial_p^v Q_0(p) = 0$$

$$p_\mu D^\mu \overline{\delta Q}(p, x) + g p^\mu F_{\mu v}(x) \partial_p^v \overline{Q}_0(p) = 0$$

$$p_\mu \mathcal{D}^\mu \delta G(p, x) - g p^\mu \mathcal{F}_{\mu v}(x) \partial_p^v G_0(p) = 0$$

Transport theory – polarization tensor

$$\delta Q(p, x) = g \int d^4 x' \Delta_p(x - x') \, p^\mu F_{\mu v}(x) \partial_p^v Q_0(p)$$

$$j^\mu[\delta Q, \delta \overline{Q}, \delta G]$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu(k) = \Pi^{\mu v}(k) A_v(k)$$

$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \overline{n}(\mathbf{p}) + 2 n_g(\mathbf{p})$$

$$\Pi^{\mu v}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E} \Big[g^{v\lambda} - \frac{p^v k^\lambda}{p^\sigma k_\sigma + i0^+} \Big] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu v}(k) = \Pi^{v\mu}(k), \quad k_\mu \Pi^{\mu v}(k) = 0$$

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Diagrammatic Hard Loop approach

$$\Pi^{\mu\nu}(k) = \left[\begin{array}{ccc} & & \end{array} \right.$$

Hard loop approximation: $k^{\mu} << p^{\mu}$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^{\mu}}{E} \left[g^{\nu\lambda} - \frac{p^{\nu} k^{\lambda}}{p^{\sigma} k_{\sigma} + i0^+} \right] \frac{\partial f(\mathbf{p})}{\partial p^{\lambda}}$$
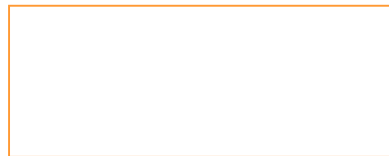
$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_{\mu}\Pi^{\mu\nu}(k) = 0$$

Chromo-hydrodynamic approach
for short time scales

Collisionless transport equation of quark distribution function $Q(p,x)$

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{ F_{\mu v}, \partial^v_p Q(p,x) \} = 0$$

$$\int dP$$

Taking into account antiquarks and gluons is straightforward

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0)\,\delta(p^2)$$

Covariant continuity

$$D_\mu n^\mu(x) = 0$$

$$n^\mu(x) \equiv \int dP\, p^\mu Q(p,x)$$

St. M. & C. Manuel, hep-ph/0606276

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{F_{\mu v}, \partial^v_p Q(p,x)\} = 0$$

$$\int dP\, p^\mu$$

$$D_\mu T^{\mu v}(x) - \frac{g}{2}\{F_{\mu v}, n^\mu(x)\} = 0$$

$$T^{\mu v}(x) \equiv \int dP\, p^\mu p^v Q(p,x)$$

$$p^2 = 0$$

$$T^\mu_\mu(x) = 0$$

Chromo-hydrodynamic equations

$$D_\mu n^\mu(x) = 0$$

$$D_\mu T^{\mu\nu}(x) - \frac{g}{2}\{F^{\mu\nu}, n_\mu(x)\} = 0$$

Postulated form of $n^\mu(x)$ and $T^{\mu\nu}(x)$:   isotropy in the local rest frame

$$n^\mu(x) = n(x)\, u^\mu(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2}\left(\varepsilon(x) + p(x)\right)\{u^\mu(x), u^\mu(x)\} - p(x)\, g^{\mu\nu}$$

$n(x)$, $\varepsilon(x)$, $p(x)$, $u^\mu(x)$ matrices! $u^\mu(x)\, u_\mu(x) = 1$

To close the system of equations: $\nabla p = 0$ or $\varepsilon = 3p \Leftarrow T_\mu^\mu = 0$

Linear response approximation

Small perturbation of the space-time homogeneous & colorless state

$$n(x) = \tilde{n} + \delta n(x), \quad \varepsilon(x) = \tilde{\varepsilon} + \delta\varepsilon(x),$$

$$p(x) = \tilde{p} + \delta p(x), \quad u^{\mu}(x) = \tilde{u}^{\mu} + \delta u^{\mu}(x)$$

$\tilde{n}, \tilde{\varepsilon}, \tilde{p}, \tilde{u}^{\mu}$ unit matrices in color space

$$\tilde{n} \gg \delta n, \quad \tilde{\varepsilon} \gg \delta\varepsilon, \quad \tilde{p} \gg \delta p, \quad \tilde{u}^{\mu} \gg \delta u^{\mu}$$

$$F^{\mu v} \sim A^{\mu} \sim \delta n$$

Solutions of the linearized equations

- $D^\mu \to \partial^\mu$ full linearization $A^\mu \sim \delta n$

- Fourier transformations   ● $\partial^\nu \delta p \approx 0$

continuity
$$k_\mu \tilde{u}^\mu \, \delta n(k) + \tilde{n} \, k_\mu \delta u^\mu(k) = 0$$

Euler
$$i(\varepsilon + \tilde{p})\tilde{u}^\mu k_\mu \delta u^\nu(k) - g\tilde{n}\tilde{u}_\mu F^{\mu\nu}(k) = 0$$

Solutions

$$\delta n(k) = ig \, \frac{\tilde{n}^2}{\varepsilon + \tilde{p}} \frac{\tilde{u}_\nu \, k_\mu}{(\tilde{u}\cdot k)^2} F^{\mu\nu}(k)$$

$$\delta u^\nu(k) = ig \, \frac{\tilde{n}}{\tilde{\varepsilon} + \tilde{p}} \frac{\tilde{u}_\mu}{\tilde{u}\cdot k} F^{\mu\nu}(k)$$

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Color current & polarization tensor

$$j^\mu(x) = -\frac{g}{2}\left(n(x)\,u^\mu(x) - \frac{1}{N_c}\mathrm{Tr}[\,n(x)\,u^\mu(x)]\right)$$

$$j^\mu(x) = \tilde{j}^\mu + \delta j^\mu(x), \qquad \boxed{\tilde{j}^\mu = 0}$$

$$\delta j^\mu(x) = -\frac{g}{2}\left(\tilde{n}\,\delta u^\mu(x) + \tilde{u}^\mu \delta n(x)\right)$$

$$\mathrm{Tr}[F^{\mu\nu}] = 0$$

polarization tensor $\qquad \Pi^{\mu\nu}(x,\,y) = -\dfrac{\delta j^\mu(x)}{\delta A_\nu(y)}$

Polarization tensor

$$\Pi^{\mu\nu}(k) = -\frac{g^2}{2}\frac{\tilde{n}^2}{\tilde{\varepsilon}+\tilde{p}}\frac{(\tilde{u}\cdot k)(\tilde{u}^\mu k^\nu + \tilde{u}^\nu k^\mu) - k^2\tilde{u}^\mu\tilde{u}^\nu - (\tilde{u}\cdot k)^2 g^{\mu\nu}}{(\tilde{u}\cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \qquad k_\mu\Pi^{\mu\nu}(k) = 0$$

From one- to multi-stream system

There are several streams in the plasma

Transport equation of quark distribution function of stream α

$$p_\mu D^\mu Q_\alpha(p,x) - \frac{g}{2} p^\mu \{ F_{\mu v}, \partial^v_p Q_\alpha(p,x) \} = 0$$

⋮

All previous steps for each stream

$$D_\mu n_\alpha^\mu(x) = 0$$

$$n_\alpha^\mu(x) = n_\alpha(x) u_\alpha^\mu(x)$$

$$D_\mu T_\alpha^{\mu v}(x) - \frac{g}{2} \{ F_\mu^{\ v}, n_\alpha^\mu(x) \} = 0$$

$$D_\mu F^{\mu v}(x) = \sum_\alpha j_\alpha^v(x)$$

Streams interact via mean field

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Polarization tensor for multi-stream system

$$\Pi^{\mu\nu}(k) = -\frac{g^2}{2}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha} \frac{(\tilde{u}_\alpha \cdot k)(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - k^2 \tilde{u}_\alpha^\mu \tilde{u}_\alpha^\nu - (\tilde{u}_\alpha \cdot k)^2 g^{\mu\nu}}{(\tilde{u}_\alpha \cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad \boxed{k_\mu \Pi^{\mu\nu}(k) = 0}$$

Connection with the kinetic theory

$$f(p) = \sum_\alpha \tilde{n}_\alpha \, \tilde{u}_\alpha^0 \delta^{(3)}(\mathbf{p} - m_\alpha \mathbf{\tilde{u}}_\alpha)$$

$$m_\alpha \equiv \frac{\tilde{\varepsilon}_\alpha + \tilde{p}_\alpha}{\tilde{n}_\alpha}$$

Effect of pressure gradients

The set of fluid equations is closed by the relation $p_\alpha(x) = \dfrac{1}{3}\varepsilon_\alpha(x)$

$$\Pi^{\mu\nu}(k) = -\frac{3g^2}{8}\sum_\alpha \frac{\tilde{n}_\alpha^2}{\tilde{\varepsilon}_\alpha}\left[\frac{(\tilde{u}_\alpha \cdot k)(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - k^2\tilde{u}_\alpha^\mu\tilde{u}_\alpha^\nu - (\tilde{u}_\alpha \cdot k)^2 g^{\mu\nu}}{(\tilde{u}_\alpha \cdot k)^2}\right.$$

$$\left. -\frac{(\tilde{u}_\alpha \cdot k)k^2(\tilde{u}_\alpha^\mu k^\nu + \tilde{u}_\alpha^\nu k^\mu) - (\tilde{u}_\alpha \cdot k)^2 k^\mu k^\nu - k^4\tilde{u}_\alpha^\mu\tilde{u}_\alpha^\nu}{k^2 + 2(\tilde{u}_\alpha \cdot k)^2}\right]$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad \boxed{k_\mu\Pi^{\mu\nu}(k) = 0}$$

Dispersion equation

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$ **chromodielectric tensor**
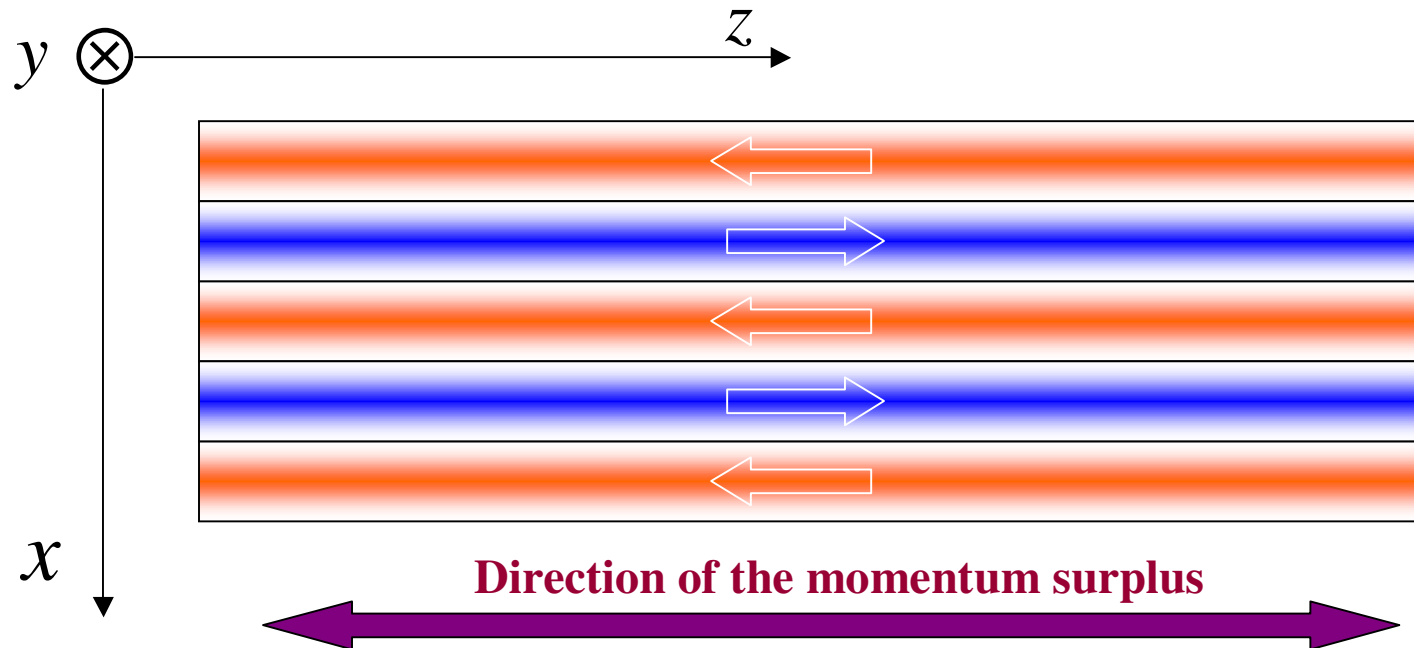
$$k^\mu \equiv (\omega, \mathbf{k})$$

Dispersion equation

$$\det[\mathbf{k}^2\delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega - \mathbf{k}\mathbf{v} + i0^+}\frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{k}\mathbf{v}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

$y \otimes$ $\xrightarrow{\hspace{5cm}}$ z

x

Direction of the momentum surplus

$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$
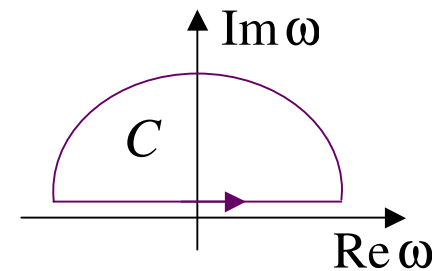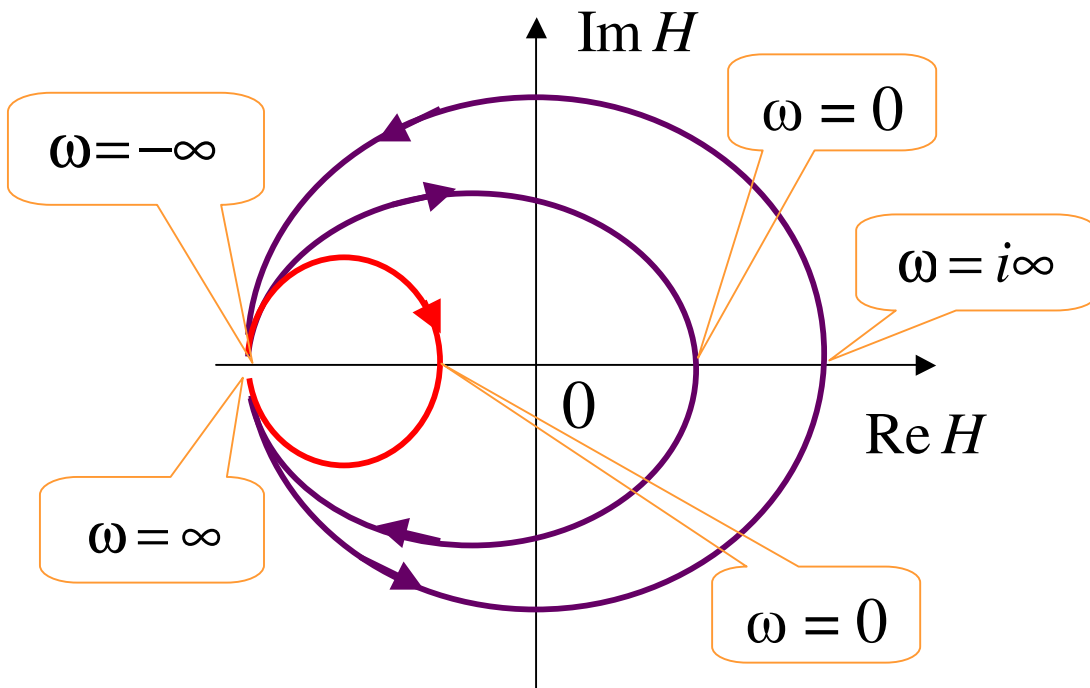
Dispersion equation

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

Existence of unstable modes – Penrose criterion

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d\ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$

Im H

$\omega = 0$

$\omega = -\infty$

$\omega = i\infty$

0

Re H

$\omega = \infty$

$\omega = 0$

Im ω

C

Re ω

There are unstable modes if

$$H(\omega = 0) < 0$$

Anisotropy!

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Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho \sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2 + \sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

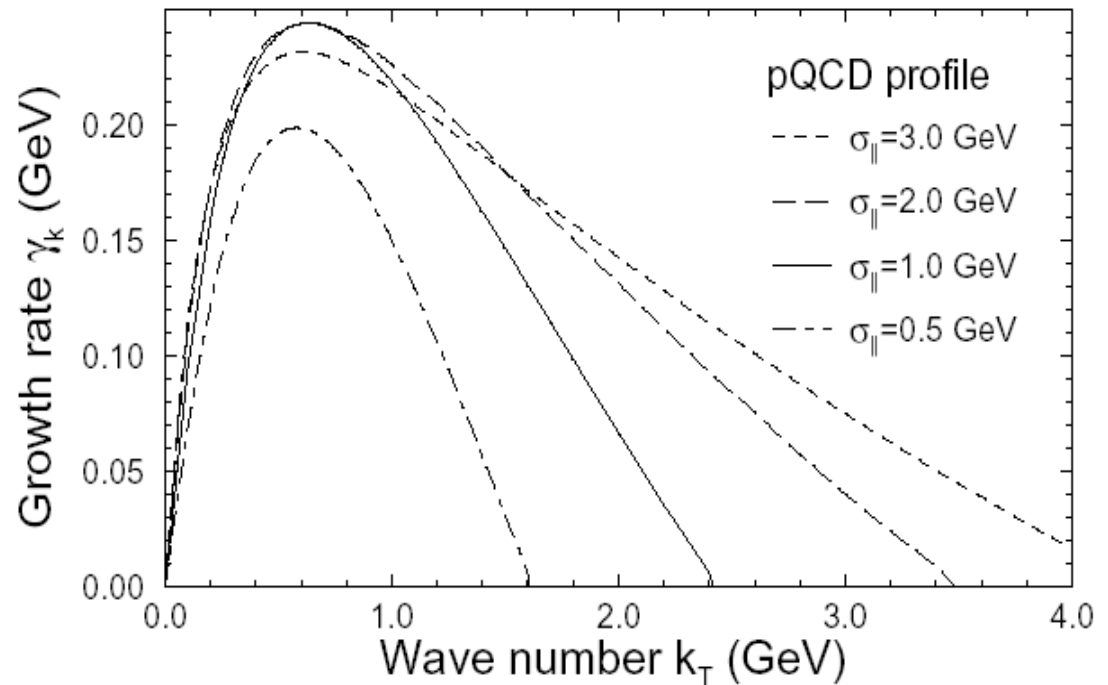$\rho = 6 \text{ fm}^{-3}$

$\alpha_s = g^2 / 4\pi = 0.3$

$\sigma_\perp = 0.3 \text{ GeV}$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i\, \gamma_k$$

$$0 < \gamma_k \in \Re$$

pQCD profile

- - - $\sigma_\parallel = 3.0$ GeV
— — $\sigma_\parallel = 2.0$ GeV
——— $\sigma_\parallel = 1.0$ GeV
— · — $\sigma_\parallel = 0.5$ GeV

Growth rate γ_k (GeV) vs. Wave number k_T (GeV)

J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\text{eff}} = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \left[f(\mathbf{p}) F^a_{\mu\nu}(x) \left(\frac{p^\nu p^\rho}{(p\cdot D)^2} \right)_{ab} F^{b\mu}_\rho(x) \right.$$

$$\left. + i\,\frac{C_F}{3}\, \tilde{f}(\mathbf{p}) \bar{\Psi}(x) \frac{p\cdot\gamma}{p\cdot D} \psi(x) \right]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics
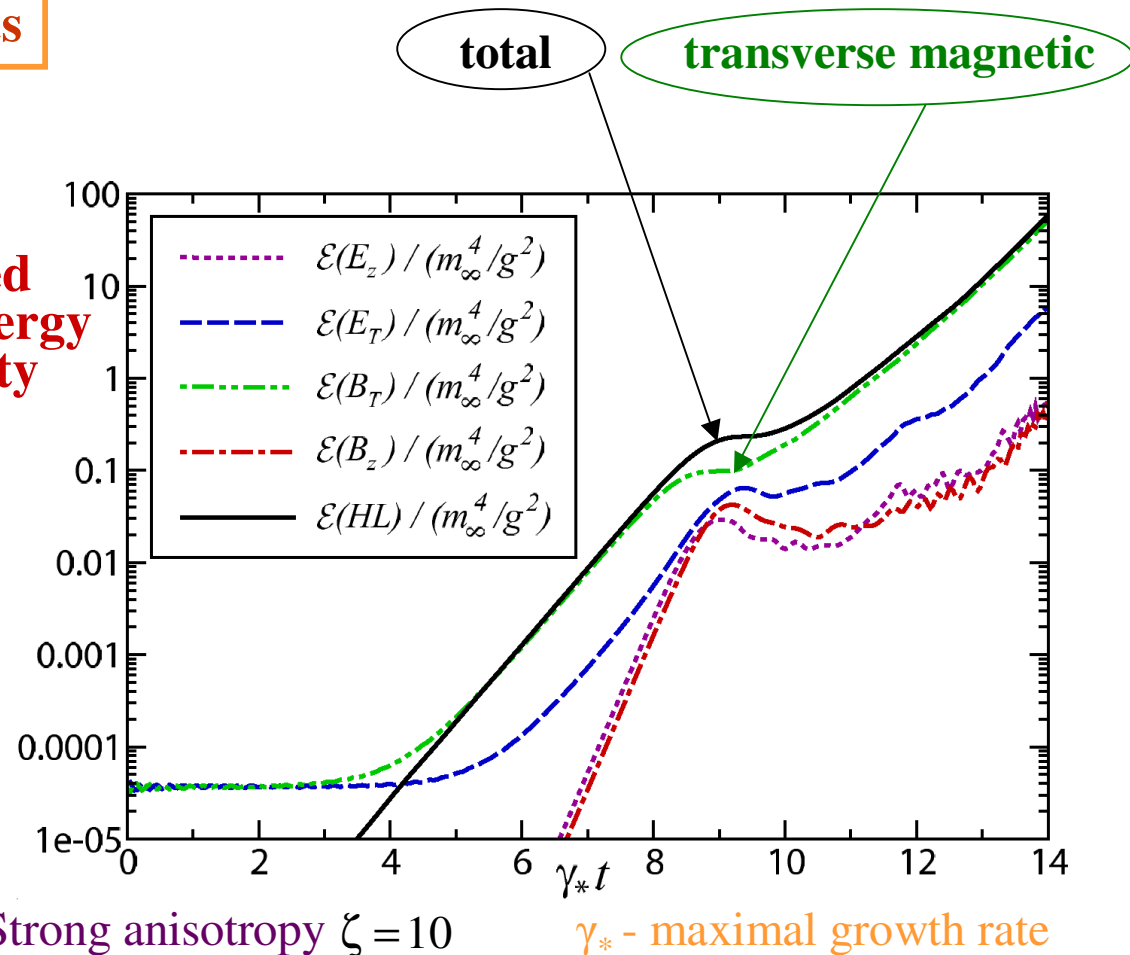
1+1 dimensions

$$A_a^\mu = A_a^\mu(t, z)$$

Scaled field energy density

Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\mathrm{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi} \int_0^\infty dp\, p^2 \frac{df_{\mathrm{iso}}(p)}{dp}$$

$$(m_D, \zeta)$$

total   **transverse magnetic**

Legend:
- $\mathcal{E}(E_z)/(m_\infty^4/g^2)$
- $\mathcal{E}(E_T)/(m_\infty^4/g^2)$
- $\mathcal{E}(B_T)/(m_\infty^4/g^2)$
- $\mathcal{E}(B_z)/(m_\infty^4/g^2)$
- $\mathcal{E}(HL)/(m_\infty^4/g^2)$

x-axis: $\gamma_* t$

Strong anisotropy $\zeta = 10$ γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

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Growth of instabilities – 1+1 numerical simulations

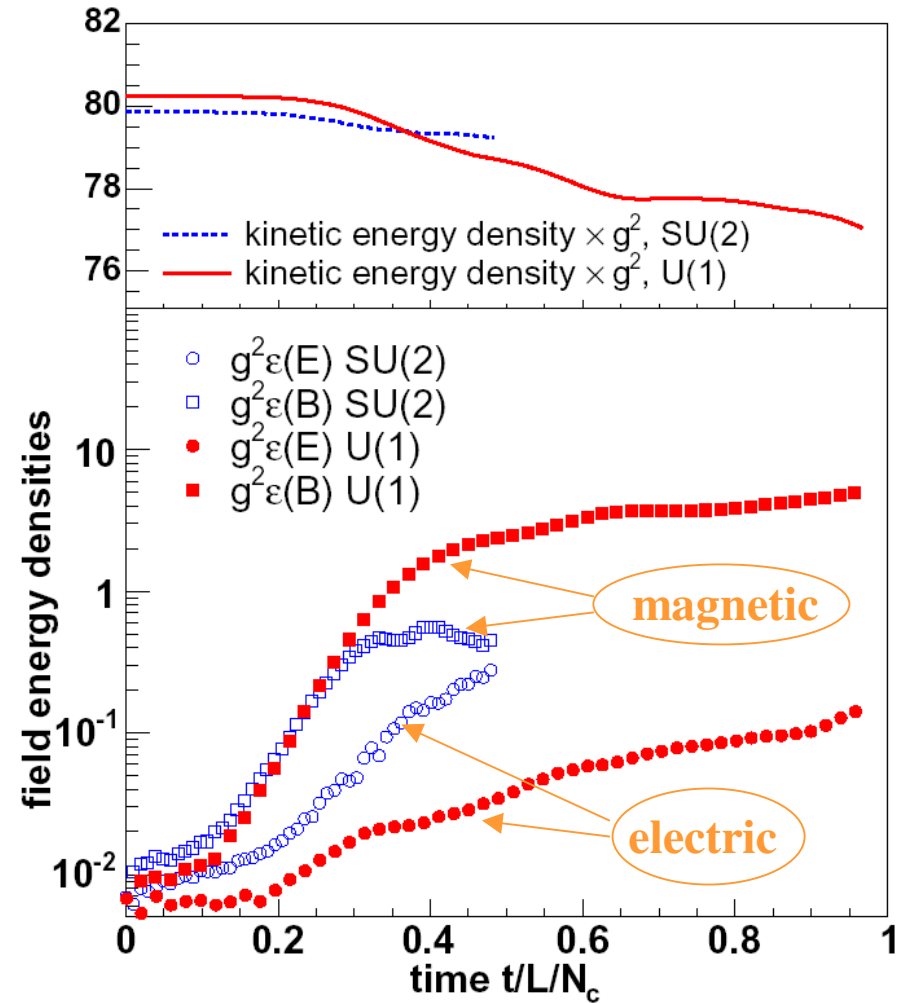**Classical system of colored particles & fields**

initial fields: Gaussian noise as in Color Glass Condensate
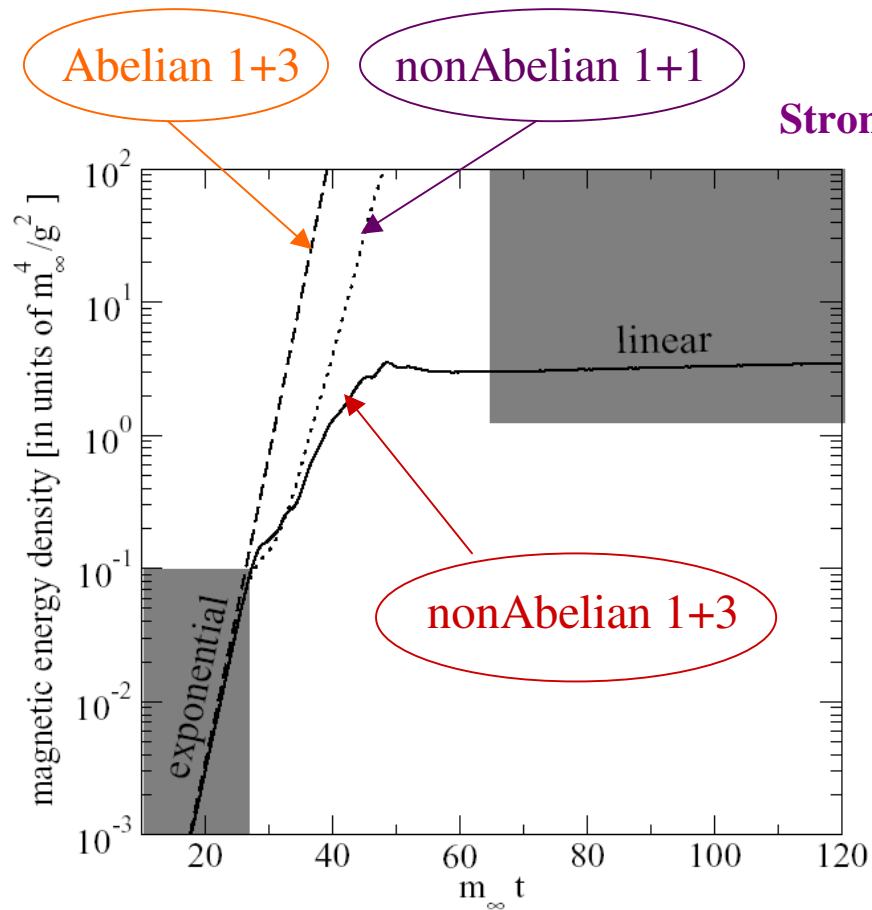
initial anisotropic particle distribution:

$$f_0(\mathbf{p},\mathbf{x}) \sim \delta(p_x)\, e^{-\frac{\sqrt{p_y^2+p_z^2}}{p_{\text{hard}}}}$$

$$p_{\text{hard}} = 10\,\text{GeV}$$

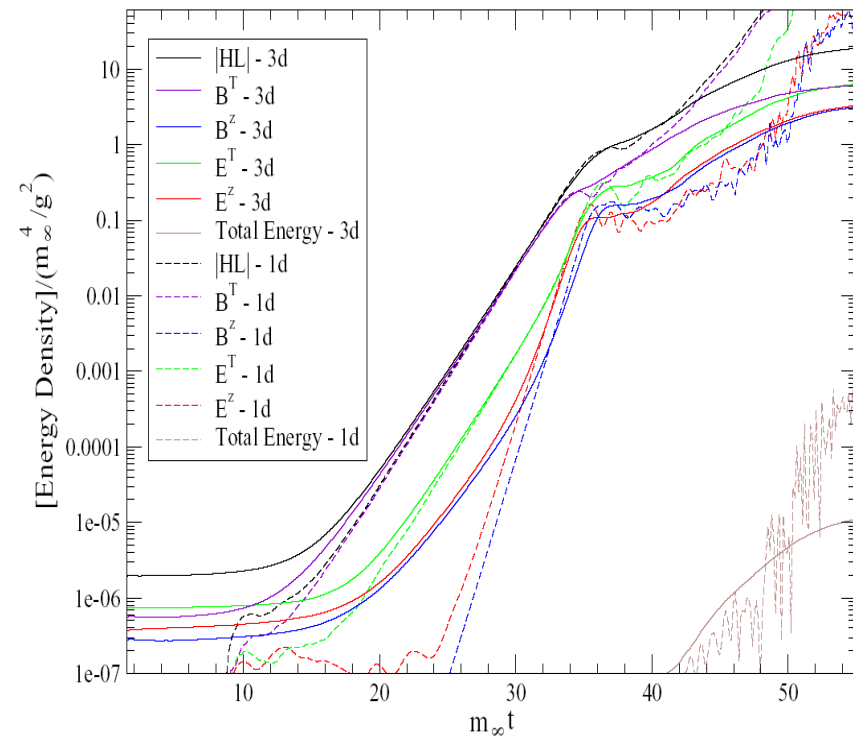$$L = 40\,\text{fm} \qquad \rho = 10\,\text{fm}^{-3}$$

A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

SU(2) Hard Loop Dynamics

Strongly anisotropic particle's momentum distribution

Abelian 1+3

nonAbelian 1+1

nonAbelian 1+3

P. Arnold, G.D. Moore & L.G. Yaffe,
Phys. Rev. **D72**, 054003 (2005)

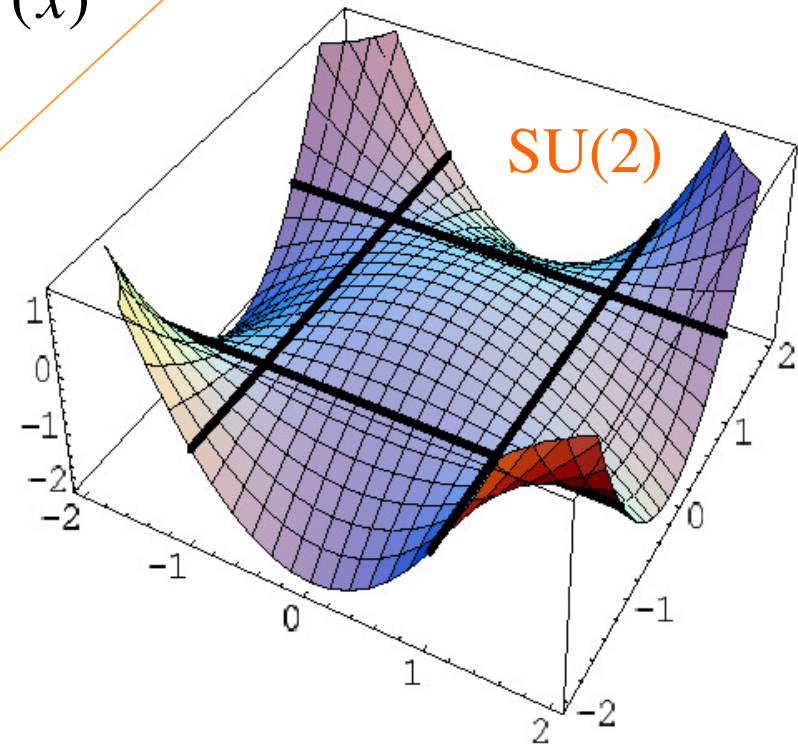A. Rebhan, P. Romatschke & M.Strickland,
JHEP **0509**, 041 (2005)

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Abelanization

$$V_{\text{eff}}[\mathbf{A}^a] = -\mu^2 \mathbf{A}^a \cdot \mathbf{A}^a + \frac{1}{4}g^2 f_{abc}f_{ade}(\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

the gauge $\quad A_0^a = 0, \quad A_i^a(t, x, y, z) = A_i^a(x)$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} = -\frac{1}{2}\mathbf{B}^a\mathbf{B}^a$$

$$= -\frac{1}{4}g^2 f_{abc}f_{ade}(\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

SU(2)

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a + \frac{g}{2}f_{abc}\mathbf{A}^b \times \mathbf{A}^c$$

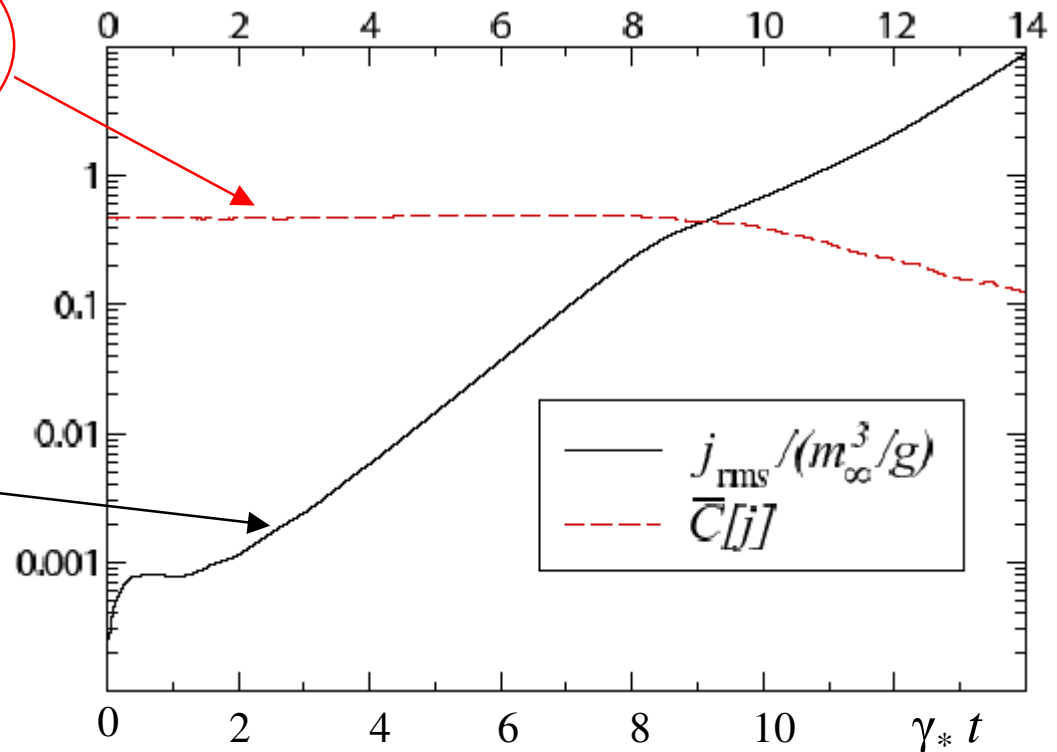P. Arnold & J. Lenaghan, Phys. Rev. D **70**, 114007 (2004)

Abelanization – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

$$\overline{C} \equiv \int_0^L \frac{dz}{L} \frac{\sqrt{\mathrm{Tr}((i[j_x, j_y])^2)}}{\mathrm{Tr}[\mathbf{j}^2]}$$

$$j_{\mathrm{rms}} \equiv \sqrt{\int_0^L \frac{dz}{L} 2\,\mathrm{Tr}[\mathbf{j}^2]}$$

Legend:
- $j_{\mathrm{rms}}/(m_\infty^3/g)$
- $\overline{C}[j]$

Horizontal axis: $\gamma_* t$

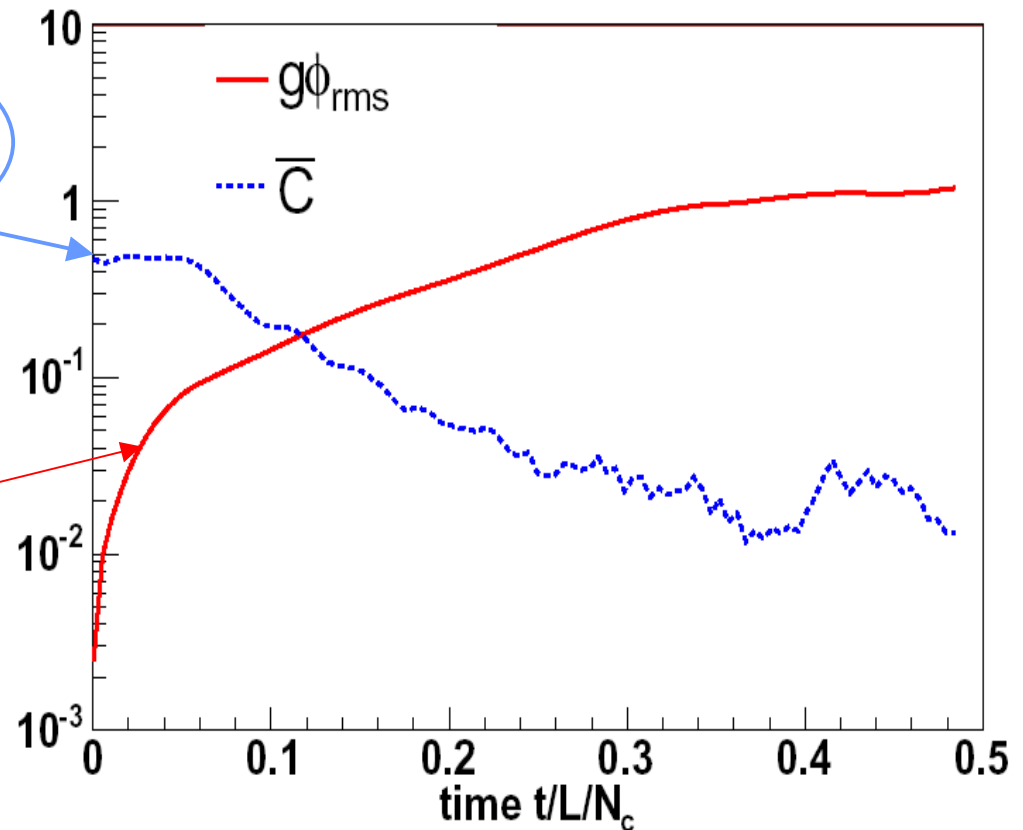A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

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Abelanization – 1+1 numerical simulations

Classical system of colored particles & fields

$$\overline{C} \equiv \int_0^L \frac{dx}{L} \frac{\sqrt{\overline{\text{Tr}((i[A_y, A_z])^2)}}}{\text{Tr}[\mathbf{A}^2]}$$

$$\phi_{\text{rms}} \equiv \sqrt{\overline{\int_0^L \frac{dx}{2L} \text{Tr}[\mathbf{A}^2]}}$$
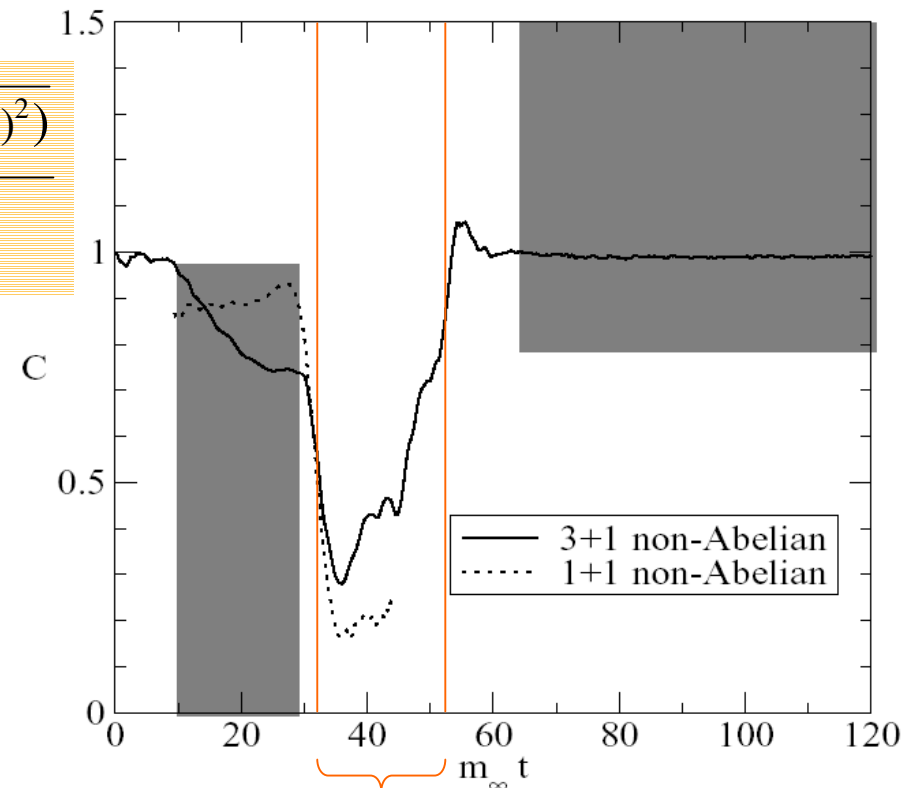
A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

Abelanization – 1+3 numerical simulations

SU(2) Hard Loop Dynamics

$$C \equiv \frac{3}{\sqrt{2}} \cdot \frac{\int \frac{d^3x}{V} \sqrt{\mathrm{Tr}((i[j_x, j_y])^2 + (i[j_y, j_z])^2 + (i[j_z, j_x])^2)}}{\int \frac{d^3x}{V} \mathrm{Tr}(\mathbf{j}^2)}$$
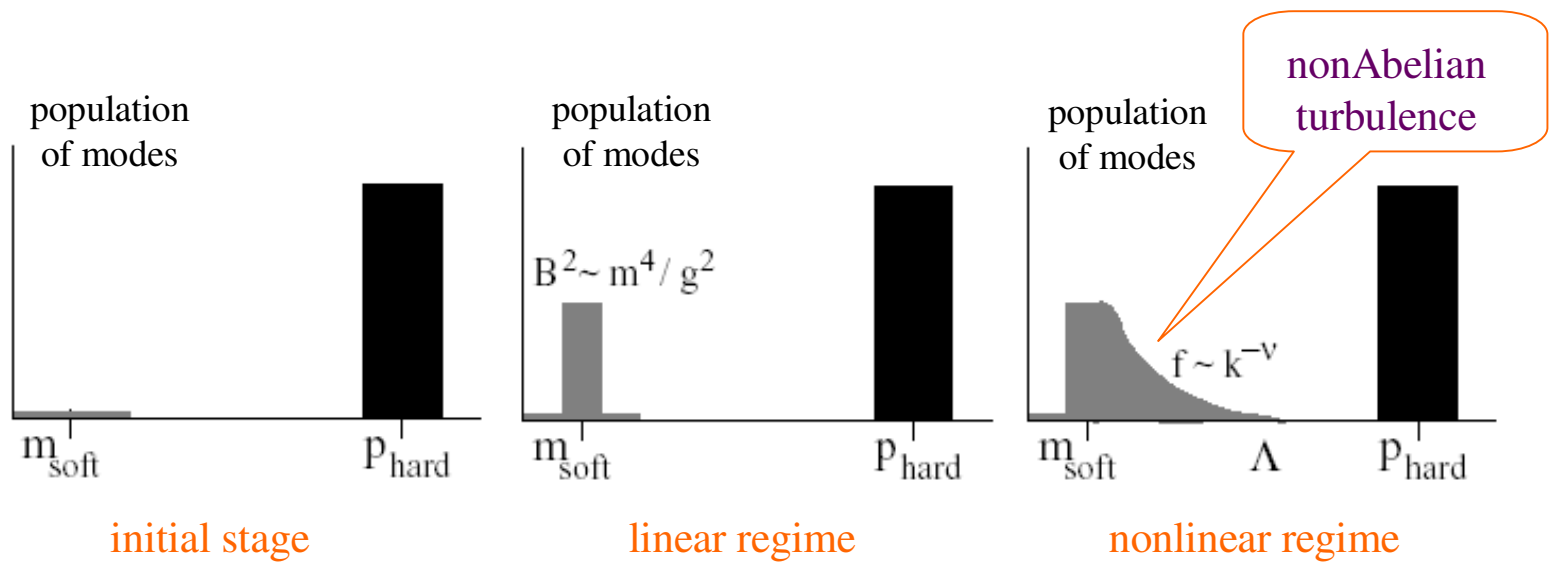
$$A_i^a \sim e^{\gamma t}$$

$$A_i^a \sim \frac{k_{\text{field}}}{g} << \frac{p_{\text{hard}}}{g}$$

NonAbelian turbulence
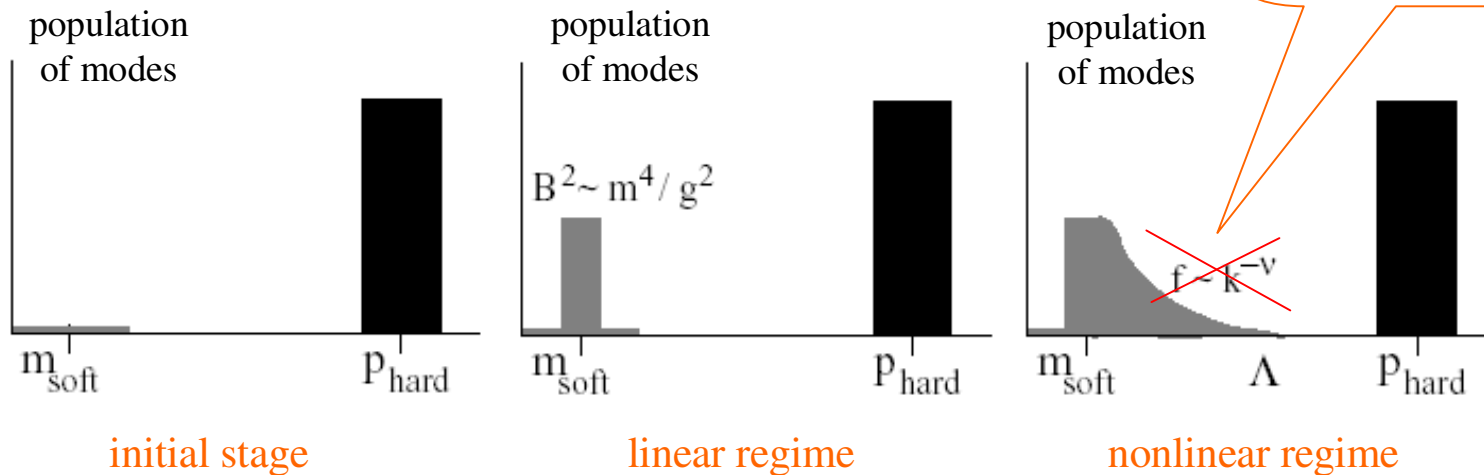
1+3 simulations of Hard Loops Dynamics

population of modes — initial stage

population of modes — linear regime, $B^2 \sim m^4/g^2$

population of modes — nonlinear regime, $f \sim k^{-\nu}$, nonAbelian turbulence

P. Arnold & G.D. Moore, Phys. Rev. **D73**, 025006 (2006);
P. Arnold & G.D. Moore, Phys. Rev. **D73**, 025013 (2006).

Ultraviolet avalanche

Classical system of colored particles & fields

Strong anisotropy

Rapid avalanche into ultraviolet

population of modes

population of modes

$B^2 \sim m^4 / g^2$

population of modes

$f \sim k^{-\nu}$

m_{soft} p_{hard}

m_{soft} p_{hard}

m_{soft} Λ p_{hard}

initial stage

linear regime

nonlinear regime

A. Dumitru, Y. Nara & M. Strickland, hep-ph/0604149

Hard Expanding Loops

fluctuation

$$Q(p,x) = Q_0(p,x) + \delta Q(p,x)$$

colorless expanding background $\quad Q_0^{ij}(p,x) = \delta^{ij} n(p,x)$

$$|Q_0(p,x)| \gg |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p,x)| \gg |\partial_p^\mu \delta Q(p,x)|$$

$$p_\mu D^\mu Q_0(p,x) = 0$$

Linearized transport equations

$$p_\mu D^\mu \delta Q(p,x) - g p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p,x) = 0$$
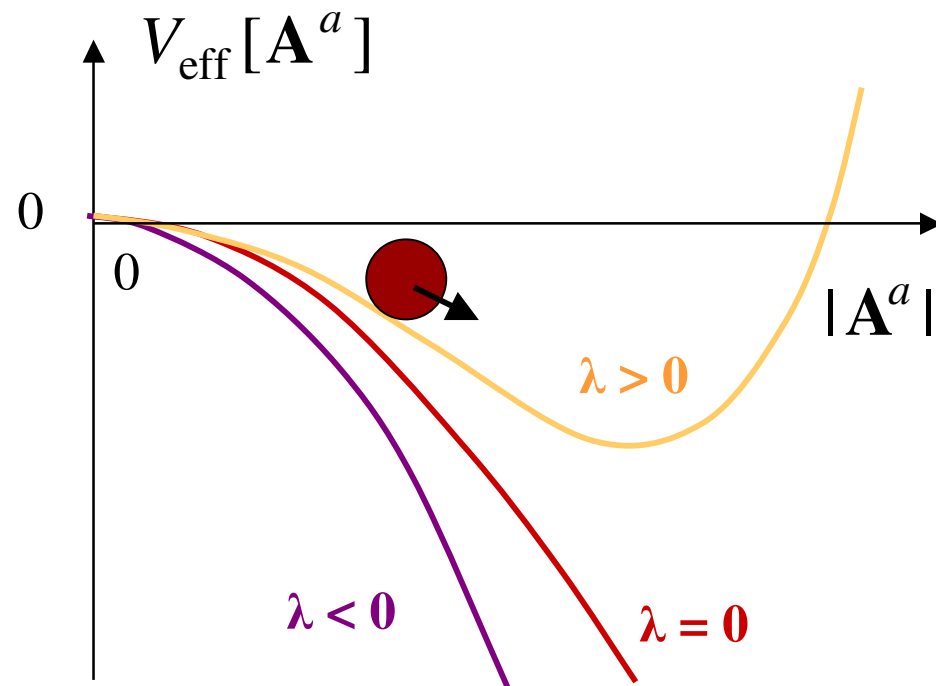
Expansion delays the onset of instability growth $\qquad A_i^a \sim e^{\lambda \sqrt{t}}$

A. Rebhan & P. Romatschke, hep-ph/0605064

Beyond Hard Loop level

$$V_{\text{eff}}[\mathbf{A}^a] = -\mu^2 \, \text{Tr}[\mathbf{A}^2] + \lambda \, \text{Tr}[\mathbf{A}^4] + \cdots$$

hard-loop term

$V_{\text{eff}}[\mathbf{A}^a]$

0

0

$|\mathbf{A}^a|$
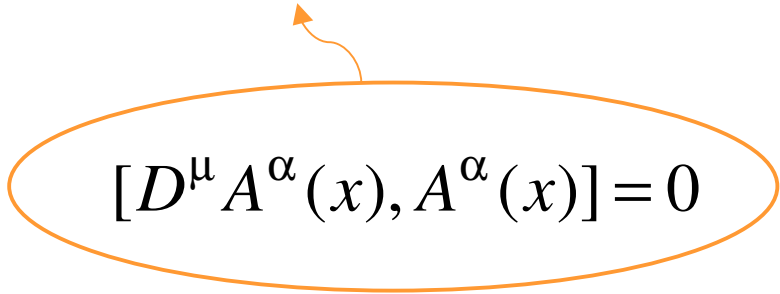
$\lambda > 0$

$\lambda < 0$

$\lambda = 0$

$\lambda = ?$

Vlasov equation

$$\left(p_\mu D^\mu - g p^\mu F_{\mu v}(x) \partial^v_p \right) Q(p,x) = 0$$

Exact solution for a system homogeneous along α direction

$$\partial^\alpha Q(p,x) = 0 = \partial^\alpha A^\mu(x)$$

$$Q(p,x) = f\left(p^\alpha - g A^\alpha(x)\right) = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left(A^\alpha(x)\right)^n \frac{\partial^n f(p^\alpha)}{\partial p_\alpha^n}$$

$$[D^\mu A^\alpha(x), A^\alpha(x)] = 0$$

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$$j^\mu[Q(x,p)] = j^\mu[f(p), A^\alpha(x)]$$

$$j^\mu(x) = -\frac{\delta S_{\text{eff}}}{A_\mu(x)}$$

$$S_{\text{eff}} \equiv -\int d^4x\, V_{\text{eff}}$$

$$V_{\text{eff}}[A^\alpha] = \sum_{n=0}^{\infty} \frac{(-g)^{n+1}}{(n+1)!} \text{Tr}[(A^\alpha)^{n+1}] \int \frac{d^3p}{(2\pi)^3} \frac{p^\alpha}{E_p} \frac{\partial^n f(p^\alpha)}{\partial p_\alpha^n}$$

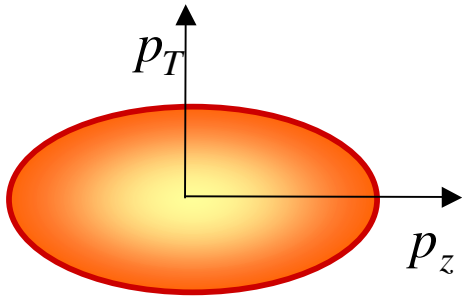# Unstable configuration of interest

$$\mathbf{j}(x) = (0,0,\, j(x)), \quad \mathbf{B}(x) = (0, B(x), 0), \quad \mathbf{A}(x) = (0,0,\, A(x))$$

$$Q(p,x) = f\left(p_0,\, p_z - gA_z\right), \qquad A_0 = 0$$

Effective potential beyond Hard Loop

$$f(p_0, p_z) \sim \exp(-\beta p_0^2 + \alpha p_z^2) = \exp\left(-\beta(p_x^2 + p_y^2) - (\beta - \alpha) p_z^2\right)$$
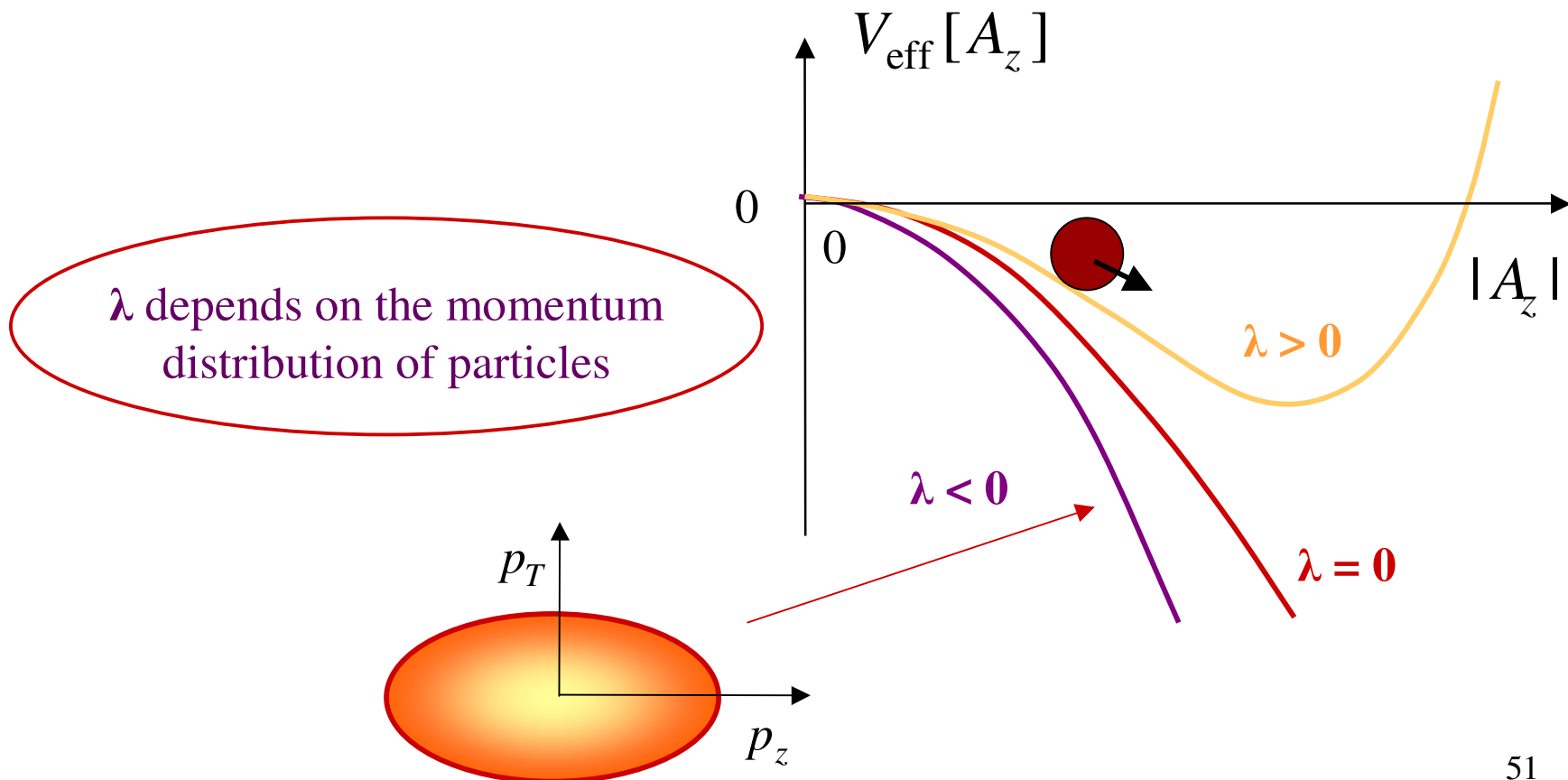
p_T

p_z

$$p_0 = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

$$V_{\text{eff}}[A_z] = -g^2 \alpha \left\langle \frac{p_z^2}{E_p} \right\rangle \text{Tr}[A_z^2]$$

$$-g^4 \left\{ \frac{1}{3} \alpha^3 \left\langle \frac{p_z^4}{E_p} \right\rangle + \frac{1}{2} \alpha^2 \left\langle \frac{p_z^2}{E_p} \right\rangle \right\} \text{Tr}[A_z^4] - \ldots$$

All terms are negative!

Effective potential beyond Hard Loop

$$V_{\text{eff}}[A_z] = -\mu^2 \text{Tr}[A_z^2] + \lambda \text{Tr}[A_z^4] + \cdots$$
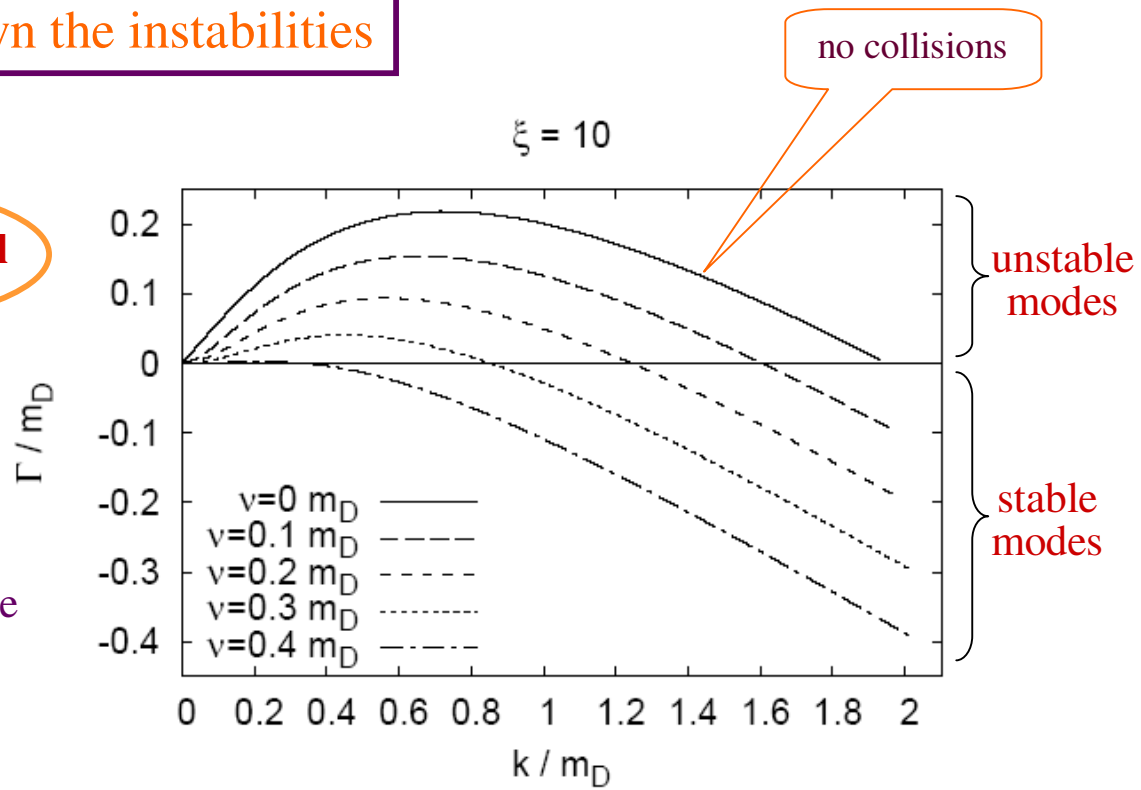
λ depends on the momentum distribution of particles

$V_{\text{eff}}[A_z]$

0

0

$|A_z|$

λ > 0

λ < 0

λ = 0

p_T

p_z

Role of collisions

Collisions slow down the instabilities

no collisions

RTE (BGK) model

$\xi = 10$

unstable modes

stable modes

Γ – instability growth rate
ν – collision frequency
m_D – Debye mass
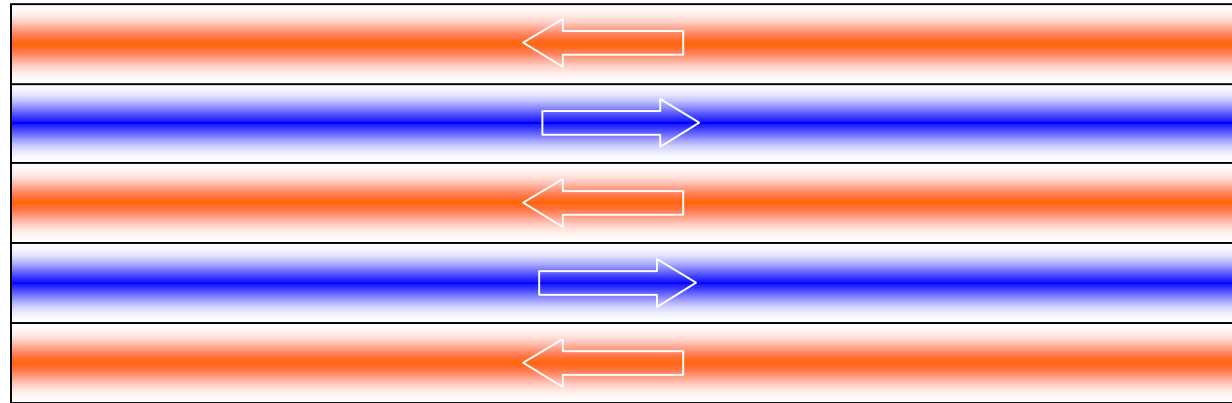ξ – anisotropy parameter

B. Schenke, M. Strickland, C. Greiner & M.H. Thoma, Phys. Rev. **D73**, 125004 (2006)
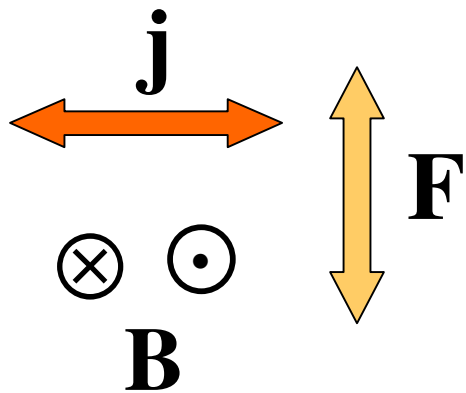
Role of collisions cont.

Instabilities do not produce entropy

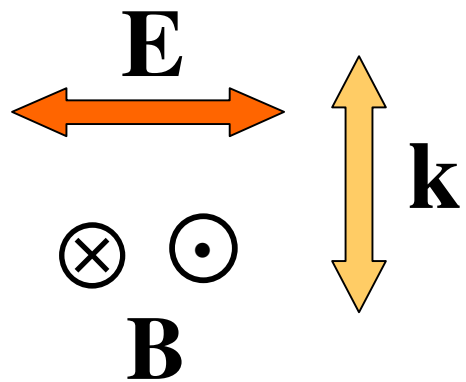Entropy is produced by collisions
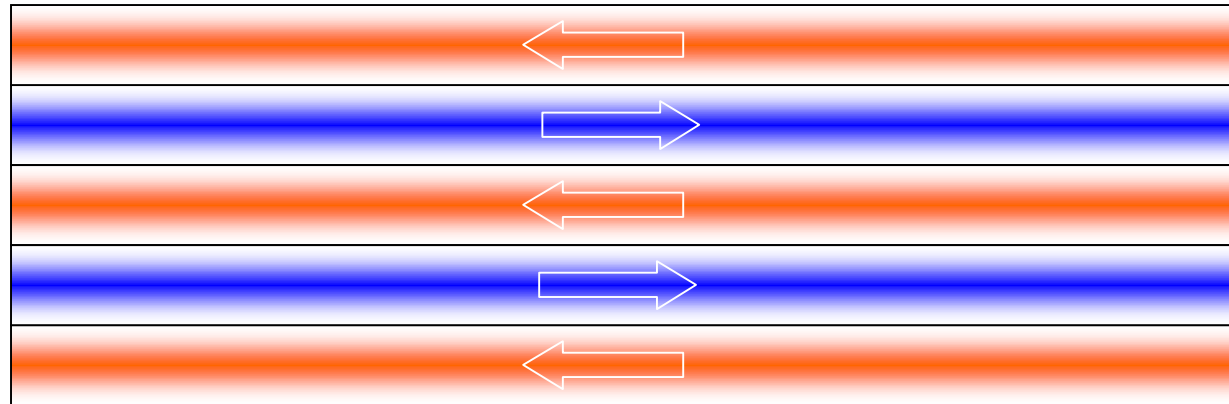
Isotropization - particles

Direction of the momentum surplus

j

F

⊗ ⊙

B

$$\Delta \mathbf{p} = \int dt\, \mathbf{F}$$

Direction of the momentum surplus

E

k

$\otimes$ $\odot$

B

$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$
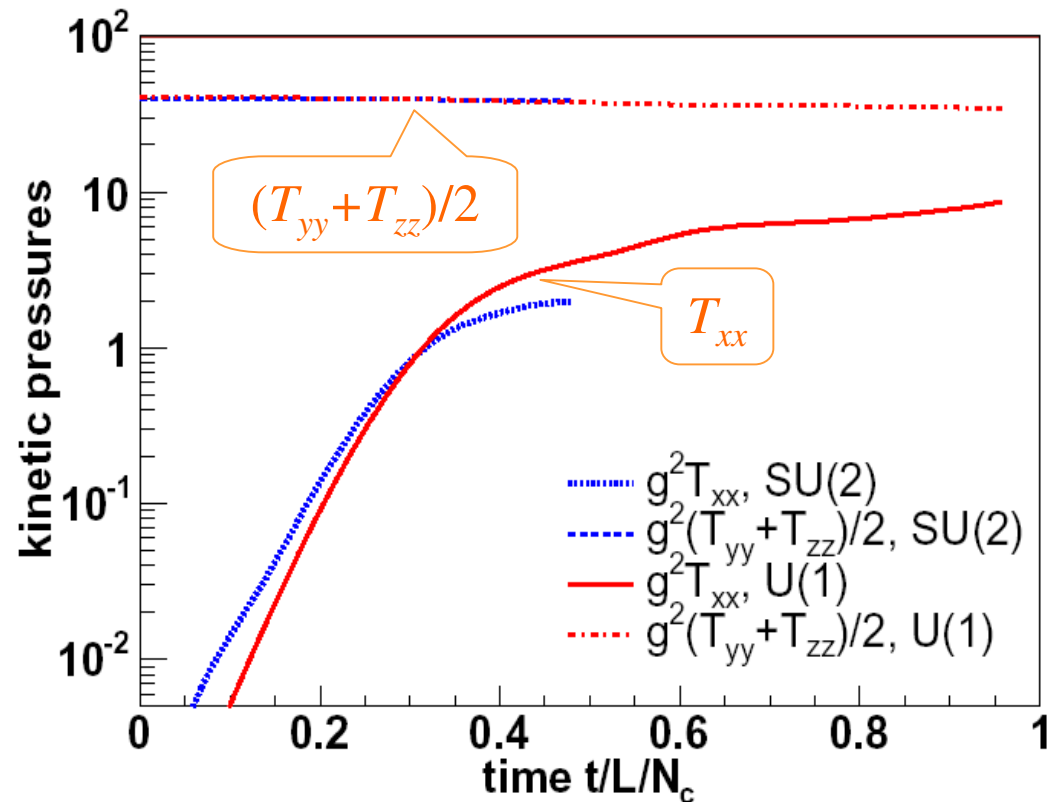
Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz}) / 2$$

A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

Instabilities in CGC & Glasma

Expansion into vacuum of self-interacting classical nonAbelian fields

Instabilities well seen!

hard modes – particles
soft modes – classical fields

Transverse magnetic modes

P. Romatschke & R. Venugopalan, Phys. Rev. Lett. **96**, 062302 (2006);
T. Lappi & L. McLerran, Nucl. Phys. **A772**, 200 (2006)

Anomalous viscosity

collisions

B field

$$\eta^{-1} = \eta_c^{-1} + \eta_a^{-1}$$

pQGP + turbulent B field $\approx$ sQGP

PM. Asakawa, S.A. Bass & B. Müller, Phys. Rev. Lett. **96**, 252301 (2006)

Conclusion

The scenario of instabilities driven equilibration
is dynamically very rich and it provides
a plausible solution of the fast equilibration
problem