Color Instabilities in the Quark-Gluon Plasma

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▶ What are the plasma instabilities?

▶ Why do the instabilities occur?

▶ Why the instabilities are important?

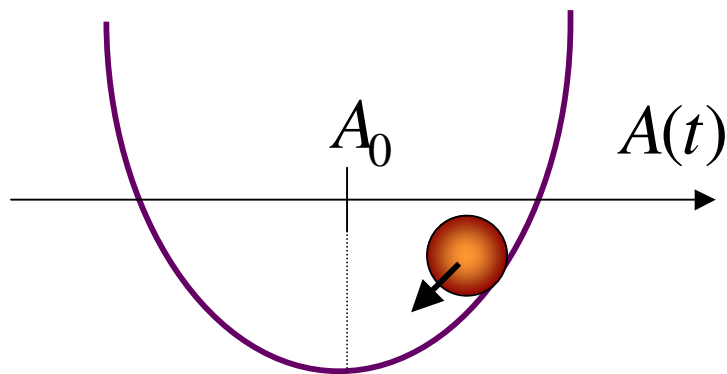# Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$
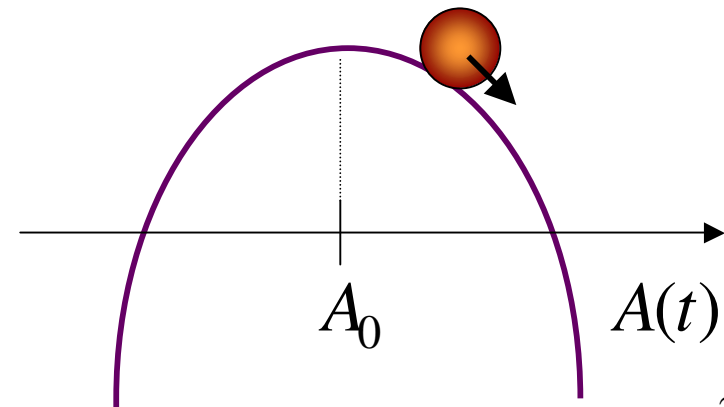
fluctuation

Instability

$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

stable configuration

unstable configuration

A_0 $A(t)$

A_0 $A(t)$

2

Terminology

Plasma instabilities – interplay of particles and classical fields

Quantum Field Theory – no particles, no classical fields

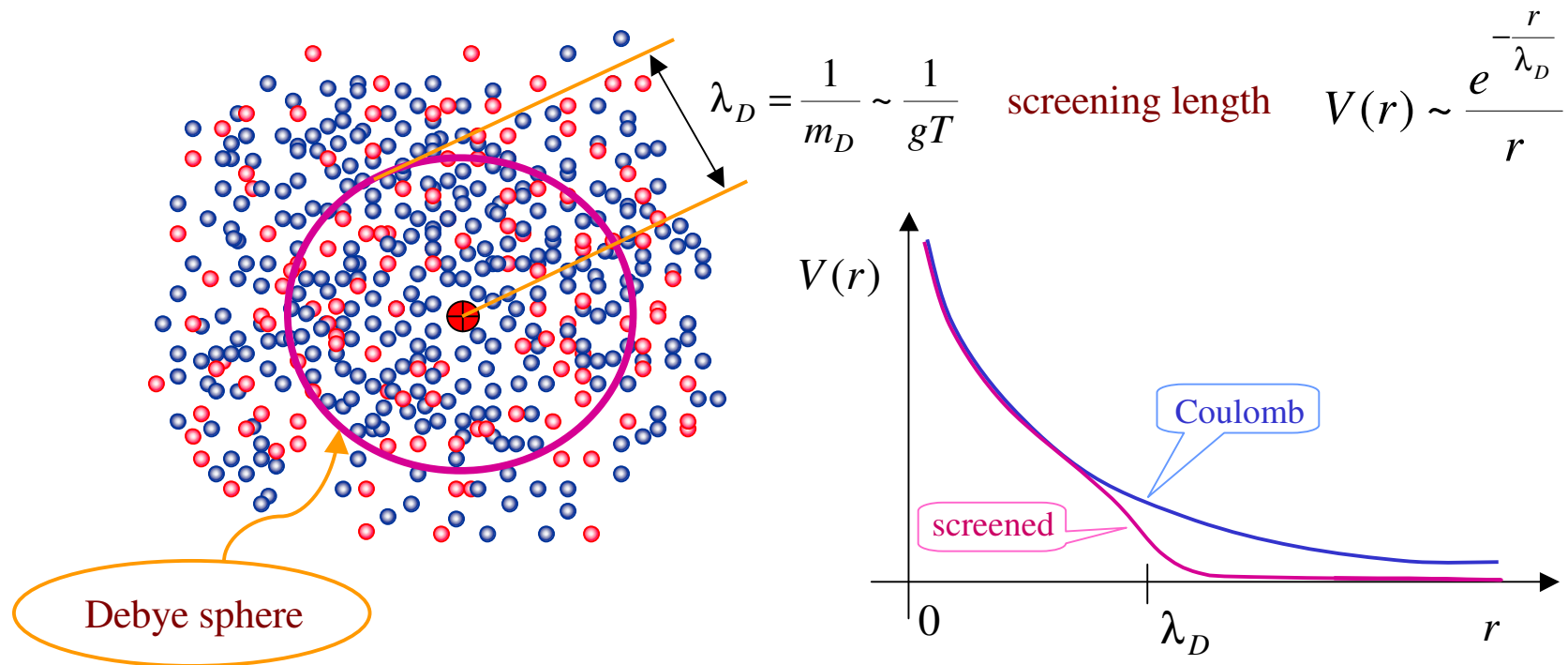$$p_{\text{hard}} \sim T$$

- particles – hard excitations, hard modes

- classical fields – highly populated soft excitations, soft modes

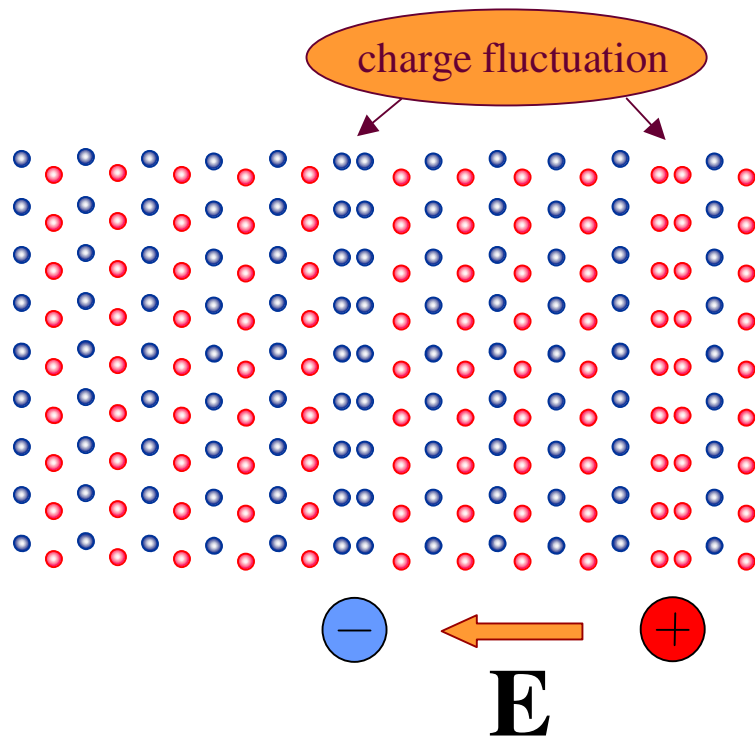$$\sim 1/g^2 \qquad p_{\text{soft}} \sim gT$$

Plasma manifests collective behavior

$$\lambda_D = \frac{1}{m_D} \sim \frac{1}{gT} \quad \text{screening length} \quad V(r) \sim \frac{e^{-\frac{r}{\lambda_D}}}{r}$$

$V(r)$

Coulomb

screened

0   λ_D r

Debye sphere

$$V_D = \frac{4}{3}\pi\lambda_D^3 \sim \frac{1}{g^3 T^3}, \qquad n \sim T^3, \qquad n V_D \sim \frac{1}{g^3} >> 1 \quad \text{if} \quad g << 1$$

In a weakly coupled plasma, there are many particles in a Debye sphere!
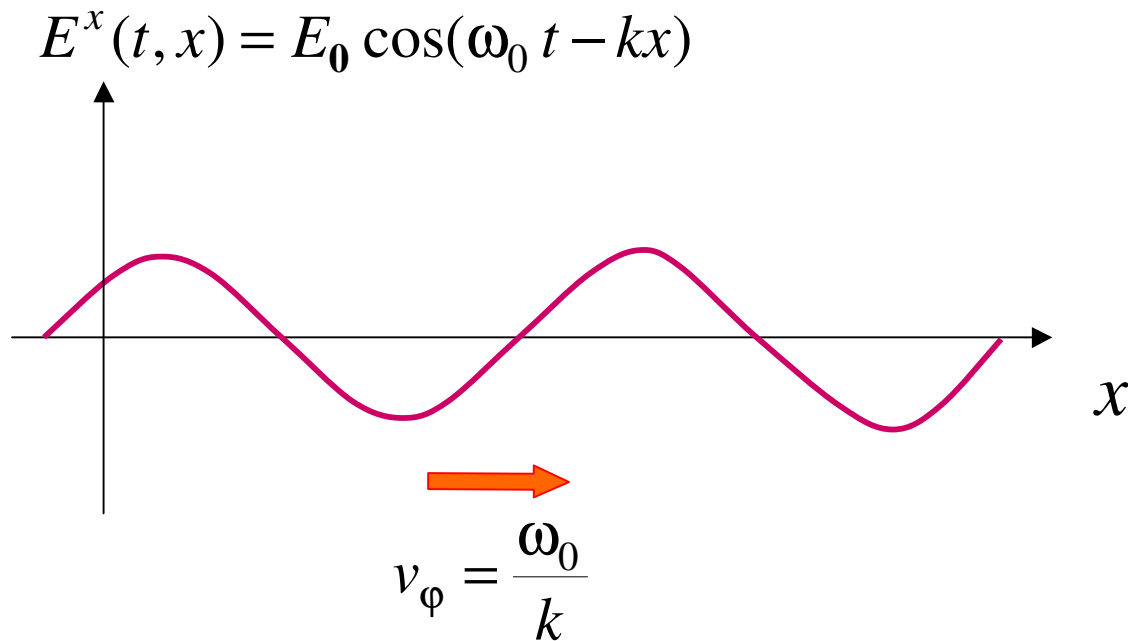
Plasma oscillations

charge fluctuation

$$\mathbf{E}(t, \mathbf{r}) = \mathbf{E_0} \cos(\omega(\mathbf{k})\, t - \mathbf{k} \cdot \mathbf{r} + \varphi)$$

$$\omega(\mathbf{k}) \approx \omega_0 \sim gT$$
$$\mathbf{k} \to 0$$

plasma frequency

E

Landau damping

$$E^x(t, x) = E_0 \cos(\omega_0 t - kx)$$

$$v_\varphi = \frac{\omega_0}{k}$$

Resonance energy transfer from electric field to particles with $v = v_\varphi$

Plasma instabilities

▶ **instabilities in configuration space –** **hydrodynamic instabilities**

▶ **instabilities in momentum space –** **kinetic instabilities**

instabilities due to non-equilibrium
momentum distribution

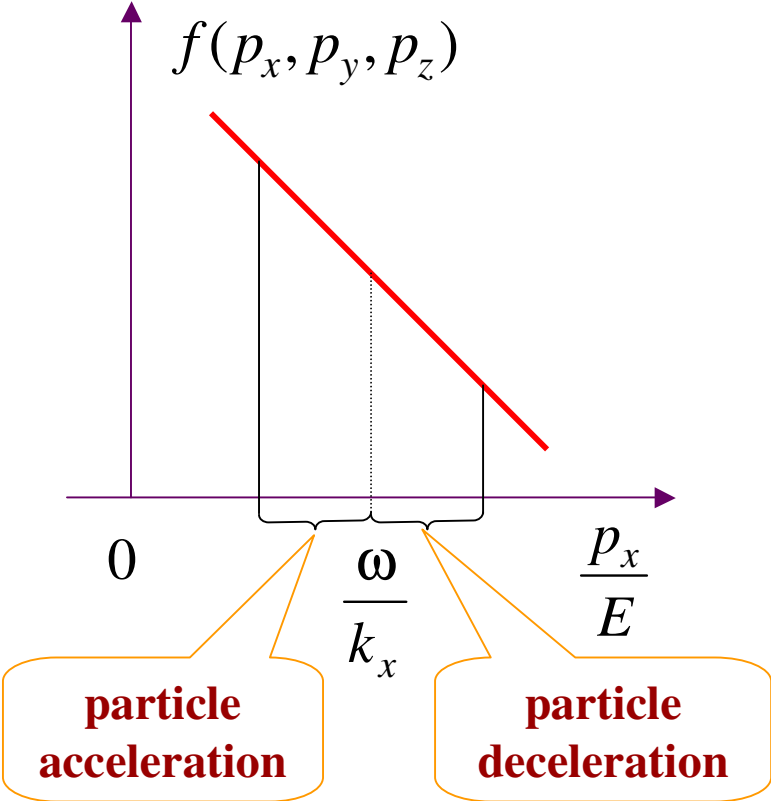$$f(\mathbf{p}) \; \text{ is } \; \text{not } \; \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

▶ **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}$, $\delta\rho \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

▶ **transverse modes** – $\mathbf{k} \perp \mathbf{E}$, $\delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$
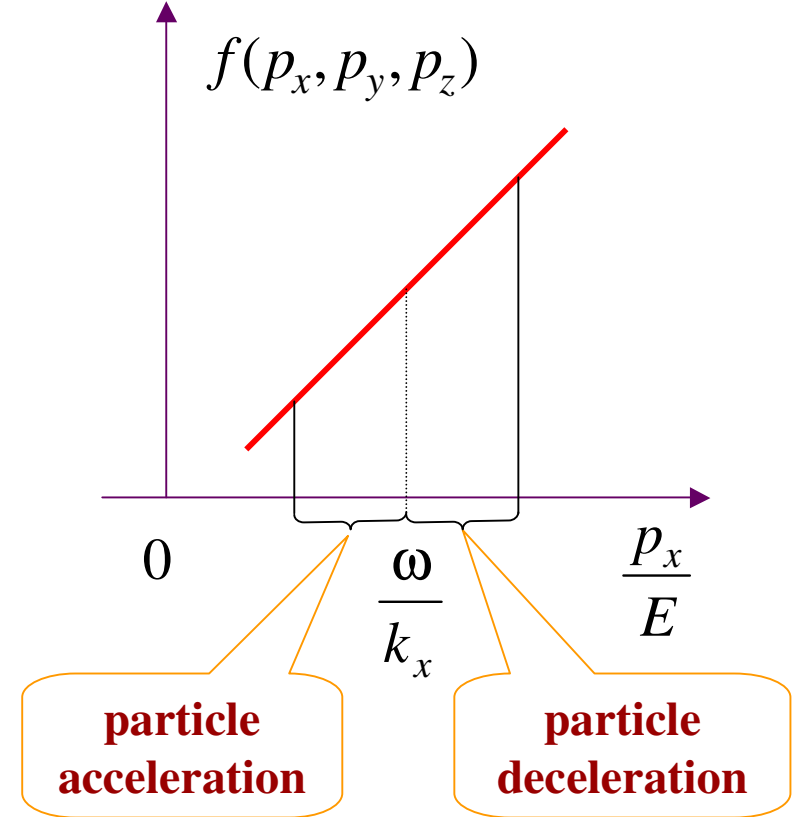
$\mathbf{E}$ – electric field, $\mathbf{k}$ – wave vector, ρ – charge density, $\mathbf{j}$ - current

Logitudinal modes

Electric field decays - damping

$f(p_x, p_y, p_z)$

$$\frac{\omega}{k_x}$$

$$\frac{p_x}{E}$$

0

particle acceleration

particle deceleration

Electric field grows - instability

$f(p_x, p_y, p_z)$

$$\frac{\omega}{k_x}$$

$$\frac{p_x}{E}$$

0

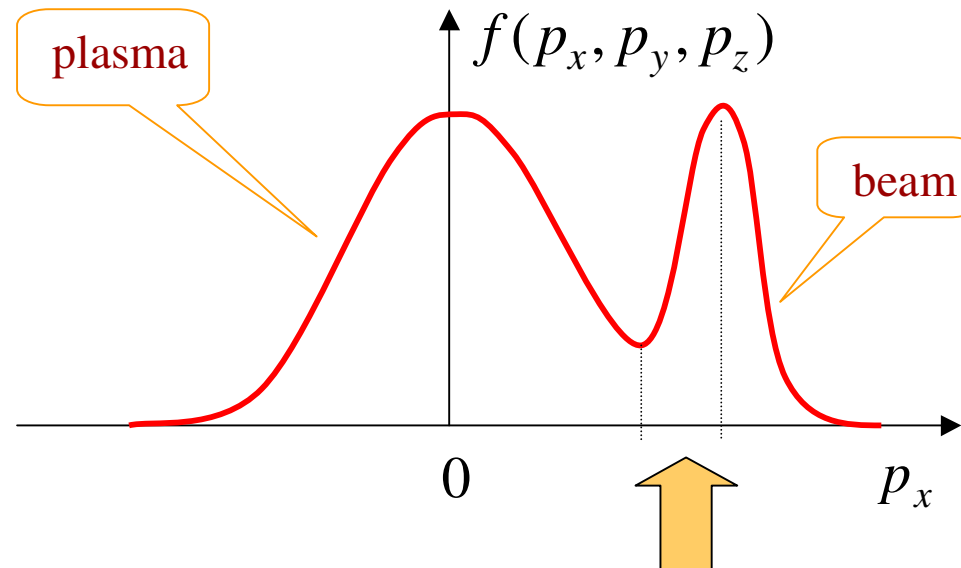**particle acceleration**

particle deceleration

$\dfrac{\omega}{k_x}$ - phase velocity of the electric field wave,

$\dfrac{p_x}{E}$ - particle's velocity
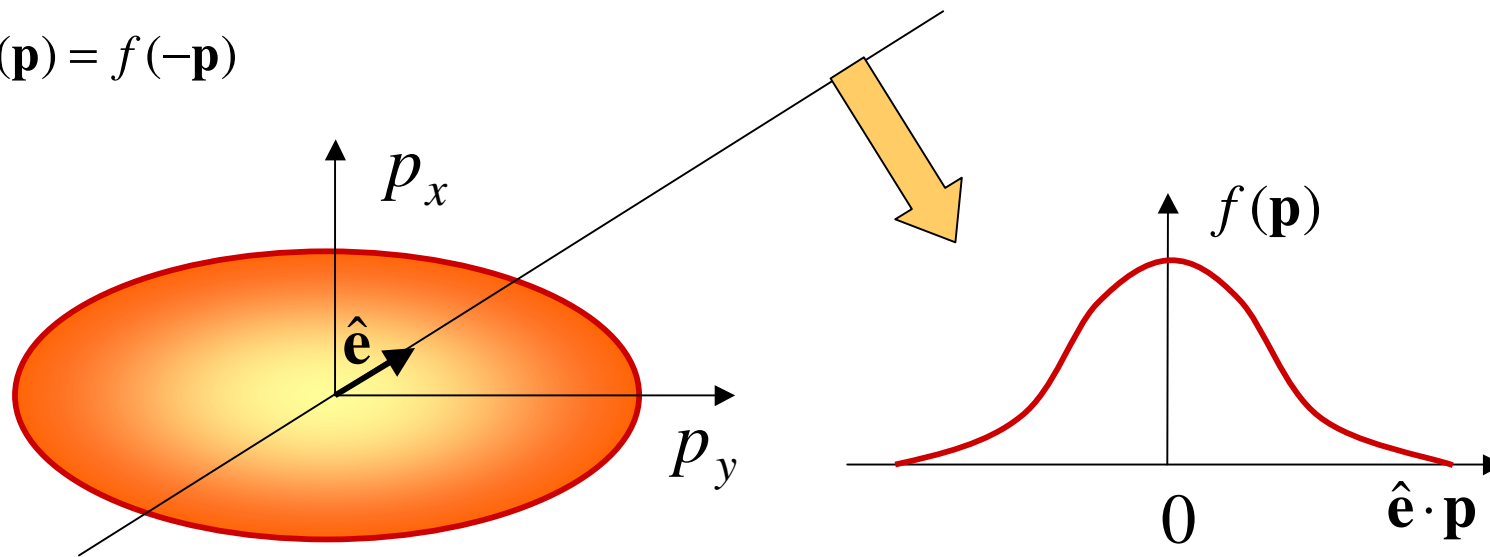
9

Logitudinal modes

unstable configuration

plasma

$f(p_x, p_y, p_z)$

beam

0

p_x

Energy is transferred from particles to fields

Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution
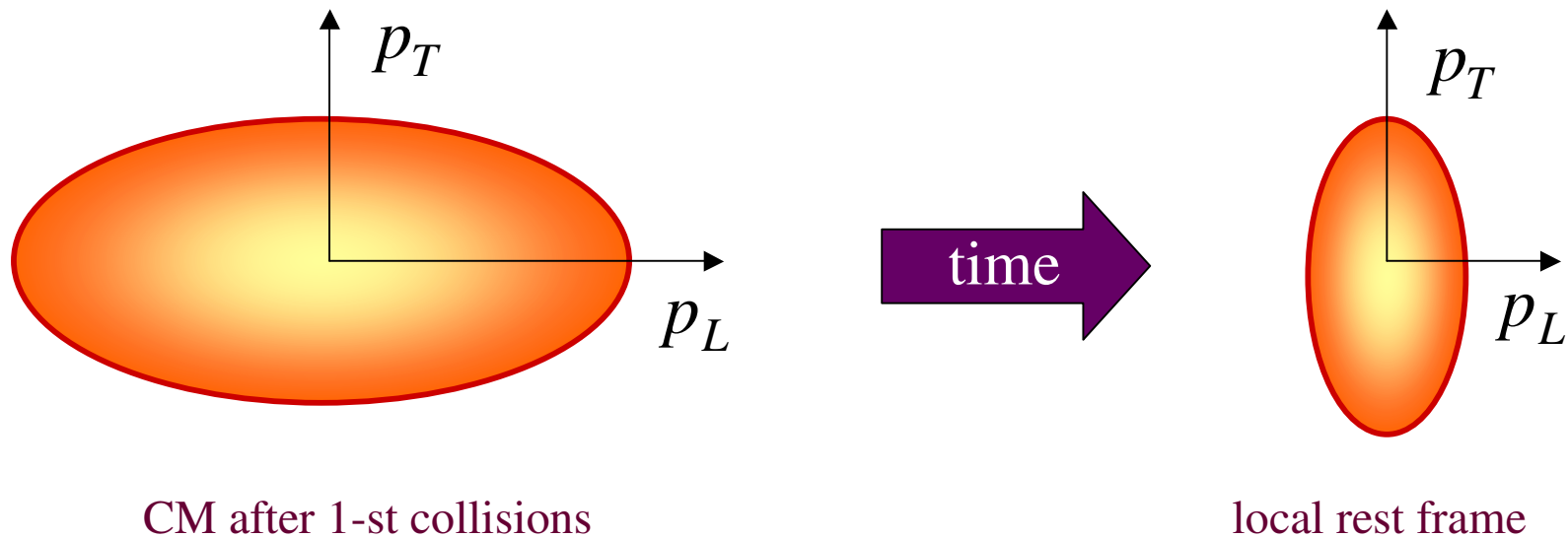
$$f(\mathbf{p}) = f(-\mathbf{p})$$

Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

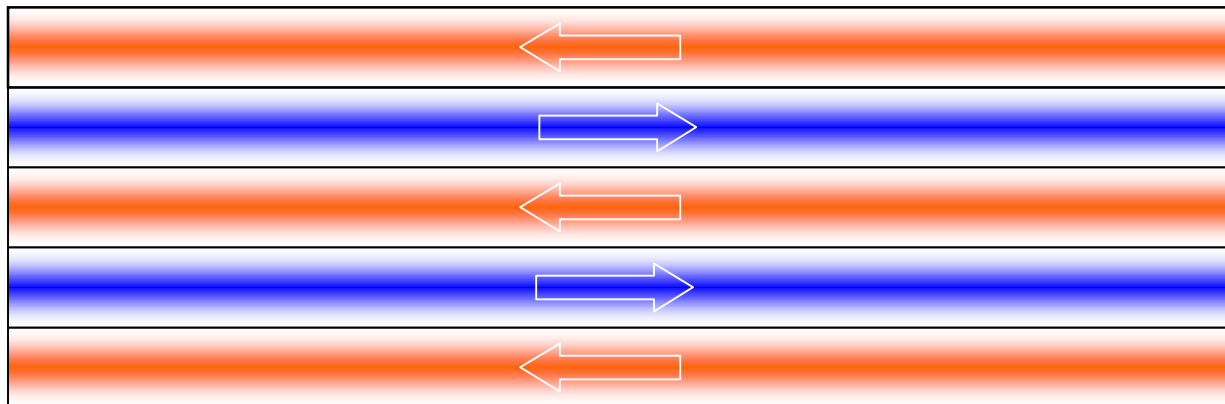**Parton momentum distribution is initially strongly anisotropic**

p_T

p_L

time

p_T

p_L

CM after 1-st collisions

local rest frame

Seeds of instability

$\left\langle j_a^\mu(x) \right\rangle = 0$ **but current fluctuations are finite**

$$\left\langle j_a^\mu(x_1)\, j_b^\nu(x_2) \right\rangle = \frac{1}{2}\delta^{ab}\int \frac{d^3 p}{(2\pi)^3}\frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{v}t) \neq 0$$
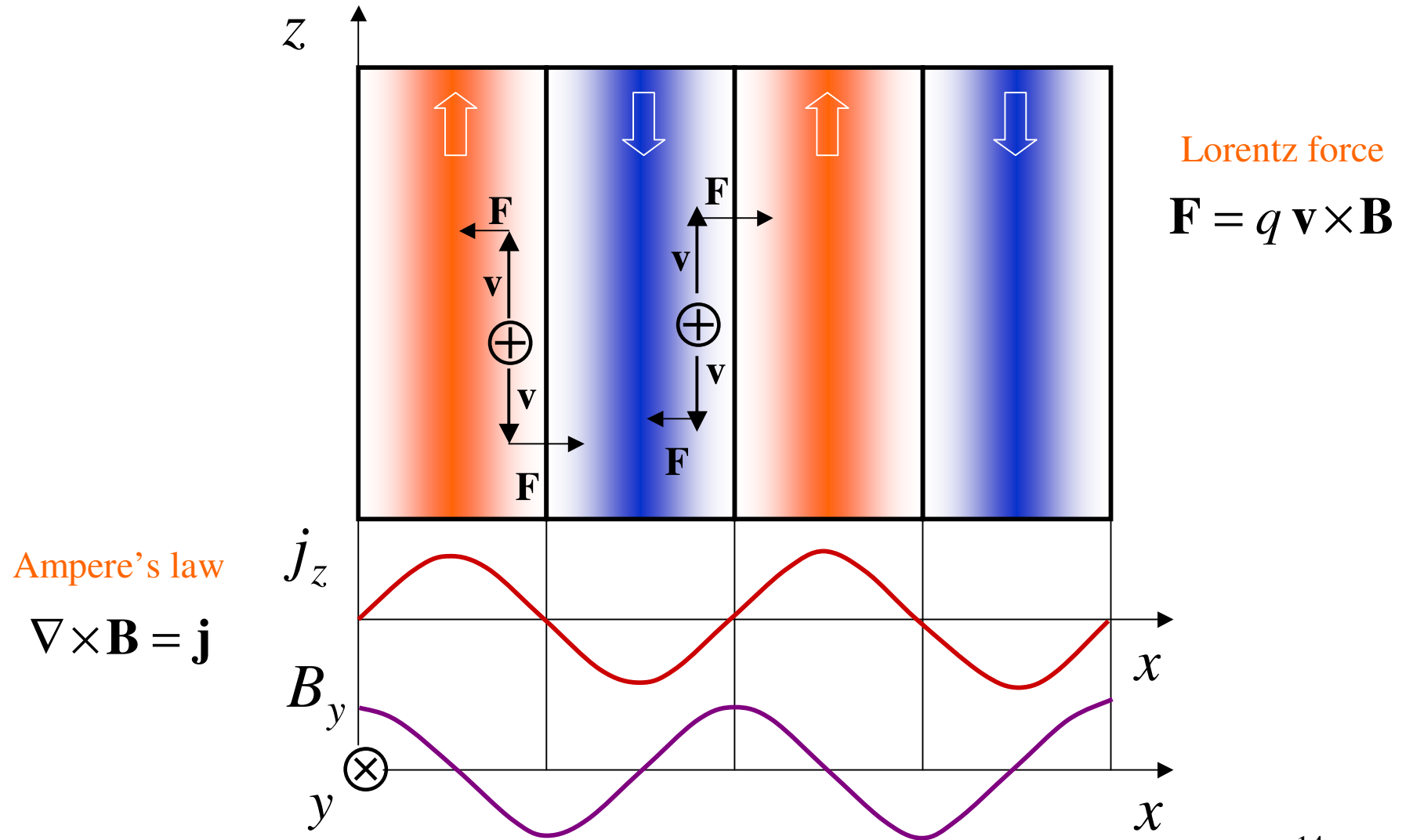
$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$

Direction of the momentum surplus

Mechanism of filamentation

Lorentz force

$$\mathbf{F} = q\,\mathbf{v}\times\mathbf{B}$$

Ampere's law

$$\nabla\times\mathbf{B} = \mathbf{j}$$

14

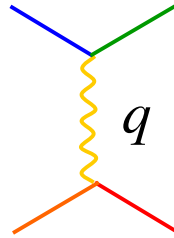# Instabilities vs. collisions

Time scale of parton-parton scattering

$$t_{hard} \sim \frac{1}{g^4 \ln(1/g) T}$$

$$t_{soft} \sim \frac{1}{g^2 \ln(1/g) T}$$

q

hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{collec} \sim \frac{1}{g\,T}$$

$$g^2 \ll 1 \implies t_{hard} \gg t_{soft} \gg t_{collec}$$

The instabilities are fast!

15

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with $\mathrm{Im}\,\omega > 0$ $\quad \Rightarrow \quad A^\mu(x) \sim e^{\mathrm{Im}\,\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. How to get it?

Transport theory – distribution functions

Distribution functions of quarks $Q(p,x)$ and antiquarks $\overline{Q}(p,x)$

are gauge dependent $N_c \times N_c$ matrices

The gauge transformation:

$$Q(p,x) \rightarrow U(x)Q(p,x)U^{-1}(x)$$

Distribution function of gluons $G(p,x)$ is $(N_c^2-1) \times (N_c^2-1)$ matrix

Transport theory – transport equations

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{ F_{\mu\nu}(x), \partial^\nu_p Q \} = C \qquad \text{quarks}$$

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{ F_{\mu\nu}(x), \partial^\nu_p \overline{Q} \} = \overline{C} \qquad \text{antiquarks}$$

adjoint

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{ \mathcal{F}_{\mu\nu}, (x) \partial^\nu_p G \} = C_g \qquad \text{gluons}$$

free streaming · mean-field force · collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]$$

mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

18

Transport theory - linearization

fluctuation

$$Q(p, x) = Q_0(p) + \delta Q(p, x)$$

stationary colorless state $\quad Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| \gg |\delta Q(p, x)|, \quad |\partial_p^{\mu} Q_0(p)| \gg |\partial_p^{\mu} \delta Q(p, x)|$$

Linearized transport equations

$$p_{\mu} D^{\mu} \delta Q(p, x) - g p^{\mu} F_{\mu v}(x) \partial_p^{v} Q_0(p) = 0$$

$$p_{\mu} D^{\mu} \delta \overline{Q}(p, x) + g p^{\mu} F_{\mu v}(x) \partial_p^{v} \overline{Q}_0(p) = 0$$

$$p_{\mu} \mathcal{D}^{\mu} \delta G(p, x) - g p^{\mu} \mathcal{F}_{\mu v}(x) \partial_p^{v} G_0(p) = 0$$

Transport theory – polarization tensor

$$\delta Q(p,x) = g \int d^4 x' \Delta_p(x-x')\, p^\mu F_{\mu v}(x) \partial_p^v Q_0(p)$$

$$j^\mu[\delta Q, \delta \overline{Q}, \delta G]$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \overline{n}(\mathbf{p}) + 2 n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E} \Big[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i 0^+} \Big] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

20

Diagrammatic Hard Loop approach

$$\Pi^{\mu\nu}(k) = \left[\quad \quad + \quad \quad + \quad \quad \right]$$

Hard loop approximation: $k^{\mu} << p^{\mu}$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^{\mu}}{E} \left[g^{\nu\lambda} - \frac{p^{\nu} k^{\lambda}}{p^{\sigma} k_{\sigma} + i0^+} \right] \frac{\partial f(\mathbf{p})}{\partial p^{\lambda}}$$
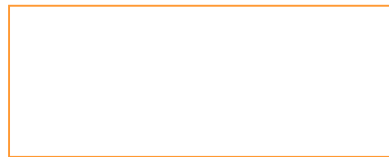
$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_{\mu}\Pi^{\mu\nu}(k) = 0$$

Chromo-hydrodynamic approach for short time scales

Collisionless transport equation of quark distribution function $Q(p,x)$

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{ F_{\mu v}, \partial_p^v Q(p,x) \} = 0$$

Taking into account antiquarks and gluons is straightforward

$$\int dP$$

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0)\, \delta(p^2)$$

Covariant continuity

$$D_\mu n^\mu(x) = 0$$

$$n^\mu(x) \equiv \int dP\, p^\mu Q(p,x)$$

Chromo-hydrodynamic approach cont.

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{ F_{\mu v}, \partial_p^v Q(p,x) \} = 0$$

$$\int dP \, p^\mu$$

$$D_\mu T^{\mu v}(x) - \frac{g}{2} \{ F_{\mu v}, n^\mu(x) \} = 0$$

$$T^{\mu v}(x) \equiv \int dP \, p^\mu p^v Q(p,x)$$

$$p^2 = 0$$

$$T_\mu^\mu(x) = 0$$

Chromo-hydrodynamic equations

$$D_\mu n^\mu(x) = 0$$

$$D_\mu T^{\mu\nu}(x) - \frac{g}{2}\{F^{\mu\nu}, n_\mu(x)\} = 0$$

Postulated form of $n^\mu(x)$ and $T^{\mu\nu}(x)$:

isotropy in the local rest frame

$$n^\mu(x) = n(x)\, u^\mu(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2}(\varepsilon(x) + p(x))\{u^\mu(x), u^\mu(x)\} - p(x)\, g^{\mu\nu}$$

$n(x),\ \varepsilon(x),\ p(x),\ u^\mu(x)$ matrices! $u^\mu(x)\, u_\mu(x) = 1$

To close the system of equations: $\nabla p = 0$ or $\varepsilon = 3p \impliedby T^\mu_\mu = 0$

Linear response approximation

Small perturbation of the space-time homogeneous & colorless state

$$n(x) = \tilde{n} + \delta n(x), \quad \varepsilon(x) = \tilde{\varepsilon} + \delta\varepsilon(x),$$

$$p(x) = \tilde{p} + \delta p(x), \quad u^{\mu}(x) = \tilde{u}^{\mu} + \delta u^{\mu}(x)$$

$\tilde{n}, \tilde{\varepsilon}, \tilde{p}, \tilde{u}^{\mu}$ unit matrices in color space

$$\tilde{n} \gg \delta n, \quad \tilde{\varepsilon} \gg \delta\varepsilon, \quad \tilde{p} \gg \delta p, \quad \tilde{u}^{\mu} \gg \delta u^{\mu}$$

$$F^{\mu\nu} \sim A^{\mu} \sim \delta n$$

Solutions of the linearized equations

- $D^\mu \rightarrow \partial^\mu$ full linearization $A^\mu \sim \delta n$

- Fourier transformations ● $\partial^\nu \delta p \approx 0$

continuity

$$k_\mu \tilde{u}^\mu \, \delta n(k) + \tilde{n} \, k_\mu \delta u^\mu(k) = 0$$

Euler

$$i(\varepsilon + \tilde{p}) \tilde{u}^\mu k_\mu \delta u^\nu(k) - g\tilde{n} \tilde{u}_\mu F^{\mu\nu}(k) = 0$$

Solutions

$$\delta n(k) = ig \, \frac{\tilde{n}^2}{\varepsilon + \tilde{p}} \, \frac{\tilde{u}_\nu \, k_\mu}{(\tilde{u} \cdot k)^2} F^{\mu\nu}(k)$$

$$\delta u^\nu(k) = ig \, \frac{\tilde{n}}{\tilde{\varepsilon} + \tilde{p}} \, \frac{\tilde{u}_\mu}{\tilde{u} \cdot k} F^{\mu\nu}(k)$$

Color current & polarization tensor

$$j^\mu(x) = -\frac{g}{2}\left(n(x)\,u^\mu(x) - \frac{1}{N_c}\mathrm{Tr}[\,n(x)\,u^\mu(x)]\right)$$

$$j^\mu(x) = \tilde{j}^\mu + \delta j^\mu(x), \qquad \boxed{\tilde{j}^\mu = 0}$$

$$\delta j^\mu(x) = -\frac{g}{2}\left(\tilde{n}\,\delta u^\mu(x) + \tilde{u}^\mu \delta n(x)\right)$$

$$\mathrm{Tr}[F^{\mu\nu}] = 0$$

polarization tensor $\qquad \Pi^{\mu\nu}(x,y) = -\dfrac{\delta j^\mu(x)}{\delta A_\nu(y)}$

Polarization tensor

$$\Pi^{\mu\nu}(k) = -\frac{g^2}{2}\frac{\tilde{n}^2}{\tilde{\varepsilon}+\tilde{p}}\frac{(\tilde{u}\cdot k)(\tilde{u}^{\mu}k^{\nu}+\tilde{u}^{\nu}k^{\mu})-k^2\tilde{u}^{\mu}\tilde{u}^{\nu}-(\tilde{u}\cdot k)^2 g^{\mu\nu}}{(\tilde{u}\cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k)\,, \qquad k_{\mu}\Pi^{\mu\nu}(k) = 0$$

Dispersion equation

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^{\mu}k^{\nu} - \Pi^{\mu\nu}(k)] = 0$$

$$k_{\mu}\Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k) \qquad \textbf{chromodielectric tensor}$$

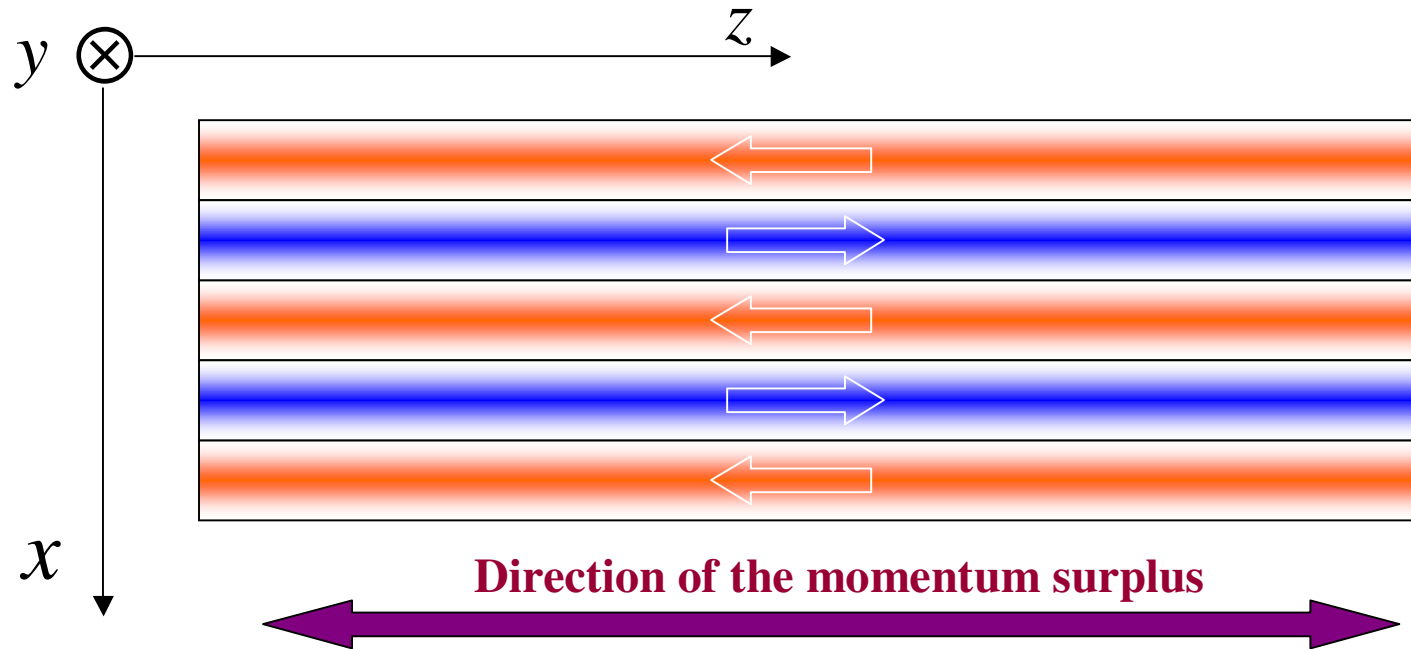$$k^{\mu} \equiv (\omega, \mathbf{k})$$

Dispersion equation

$$\det[\mathbf{k}^2\delta^{ij} - k^i k^j - \omega^2\varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega - \mathbf{k}\mathbf{v} + i0^+}\frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{k}\mathbf{v}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E \qquad 29$$

Dispersion equation – configuration of interest

$y \otimes$ — z

x

Direction of the momentum surplus

$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$
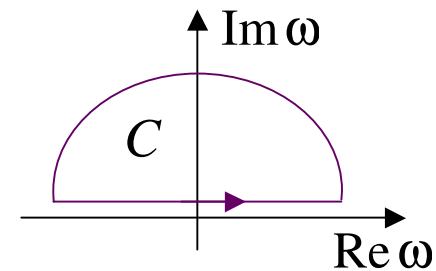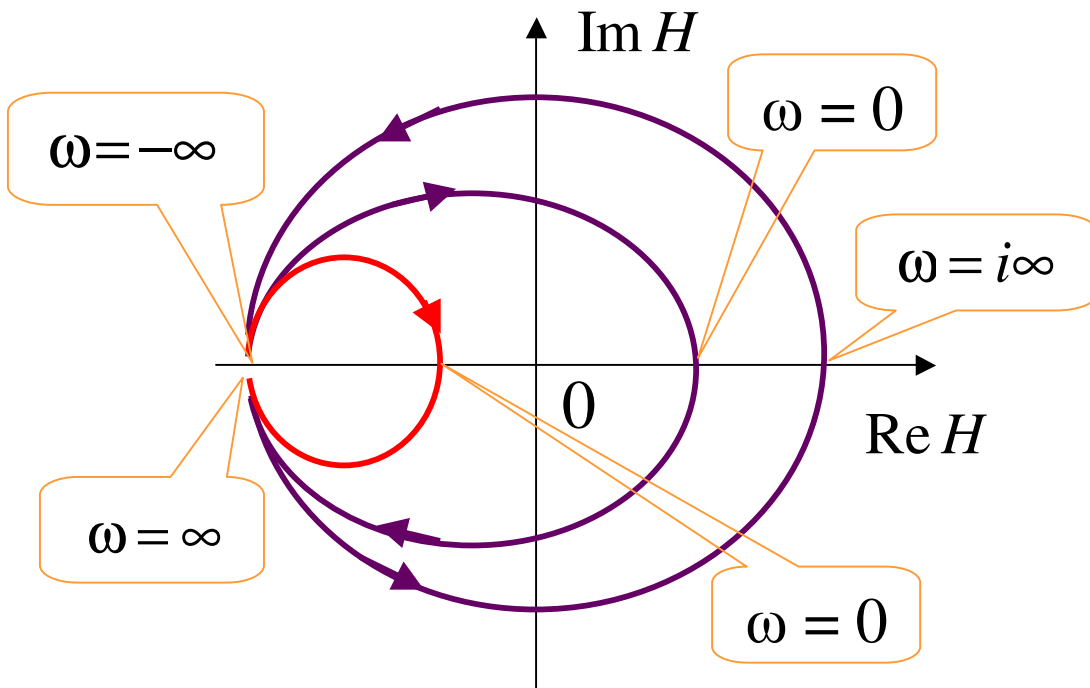
Dispersion equation

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

Existence of unstable modes – Penrose criterion

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega) \Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$

There are unstable modes if

$$H(\omega = 0) < 0$$

Anisotropy!

31

Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho \sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2 + \sigma_\perp^2)^3} \, e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

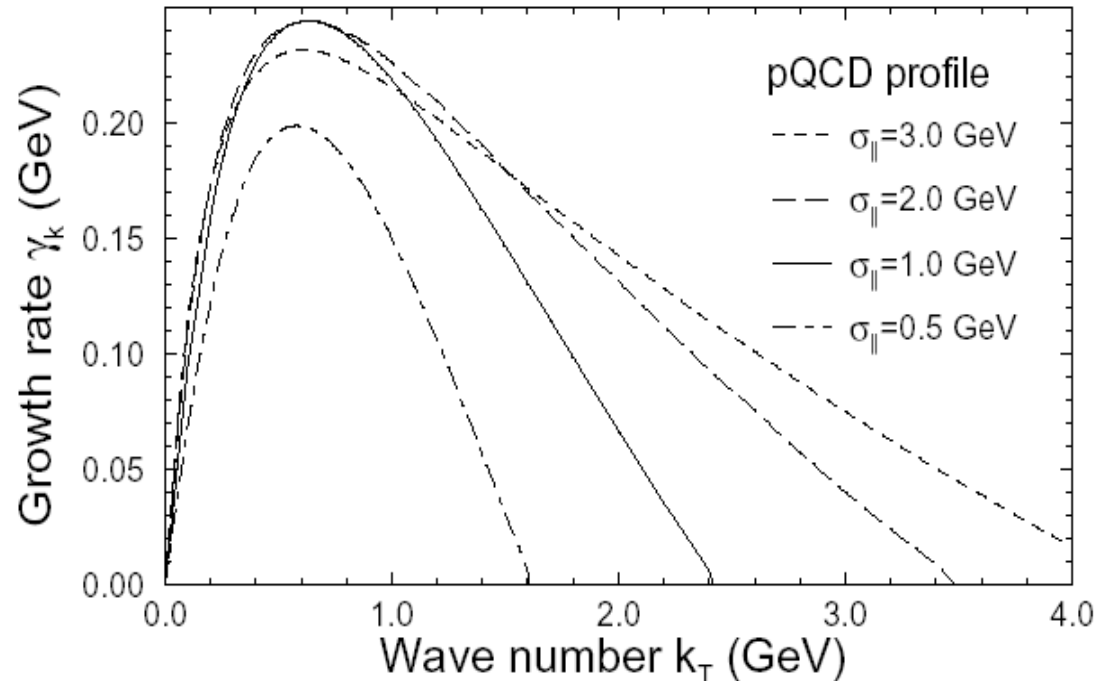$\rho = 6 \text{ fm}^{-3}$

$\alpha_s = g^2 / 4\pi = 0.3$

$\sigma_\perp = 0.3 \text{ GeV}$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i \, \gamma_k$$

$$0 < \gamma_k \in \Re$$

pQCD profile

- - - $\sigma_\parallel = 3.0$ GeV
— — $\sigma_\parallel = 2.0$ GeV
—— $\sigma_\parallel = 1.0$ GeV
– – $\sigma_\parallel = 0.5$ GeV

Growth rate γ_k (GeV) vs Wave number k_T (GeV)

J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\text{eff}} = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \left[f(\mathbf{p}) F_{\mu\nu}^a(x) \left(\frac{p^\nu p^\rho}{(p \cdot D)^2} \right)_{ab} F_\rho^{b\mu}(x) \right.$$

$$\left. + i \frac{C_F}{3} \tilde{f}(\mathbf{p}) \bar{\psi}(x) \frac{p \cdot \gamma}{p \cdot D} \psi(x) \right]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

1+1 dimensions

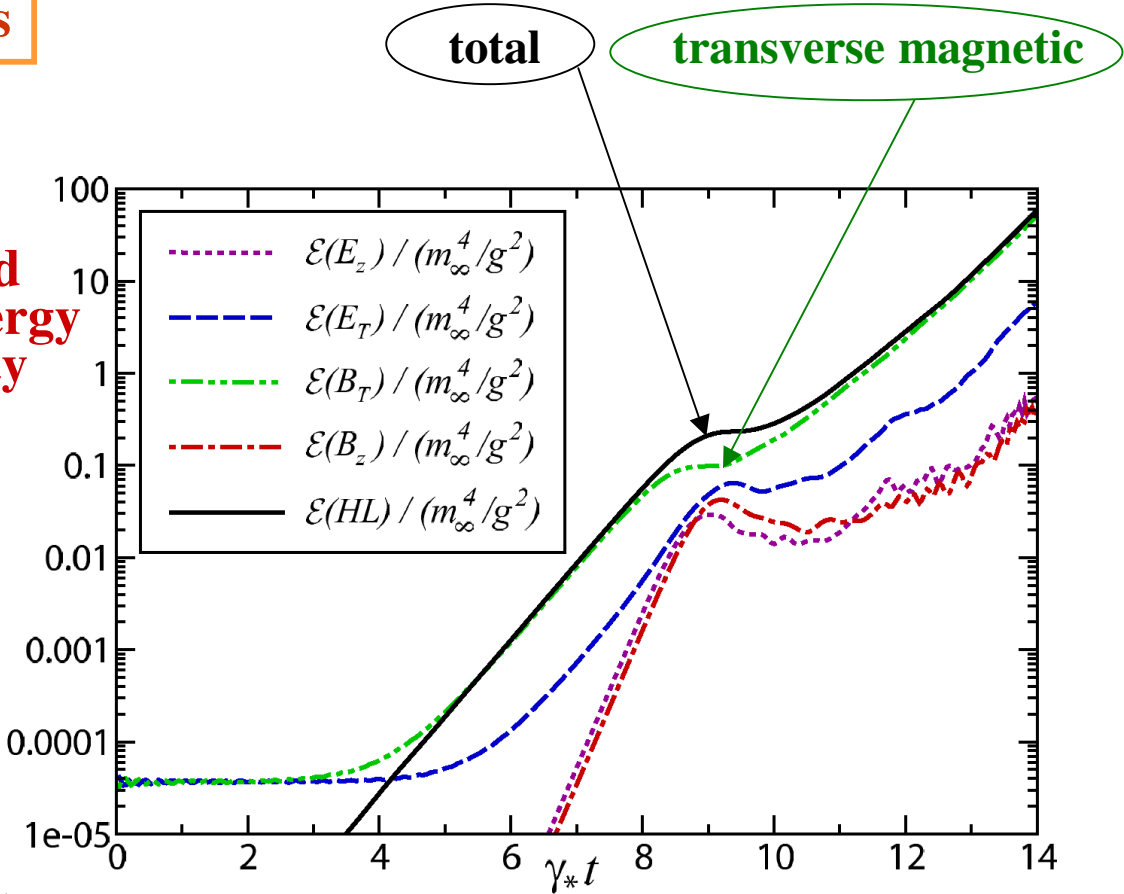$$A_a^\mu = A_a^\mu(t, z)$$

Scaled field energy density

Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

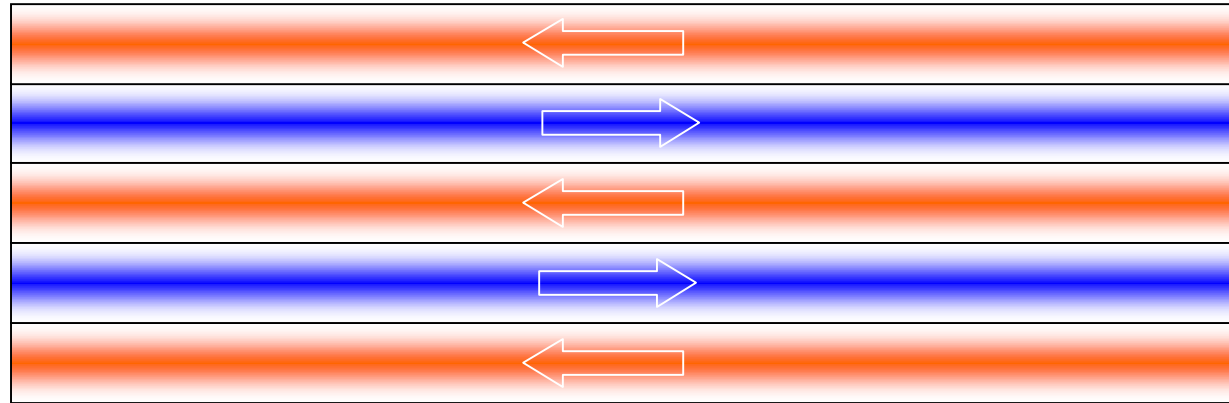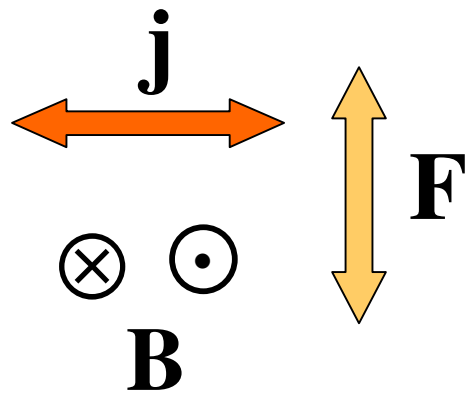$$m_D^2 = -\frac{\alpha_s}{\pi} \int_0^\infty dp\, p^2 \frac{df_{\text{iso}}(p)}{dp}$$

$$(m_D, \zeta)$$

total

transverse magnetic

Legend:
- $\mathcal{E}(E_z)/(m_\infty^4/g^2)$
- $\mathcal{E}(E_T)/(m_\infty^4/g^2)$
- $\mathcal{E}(B_T)/(m_\infty^4/g^2)$
- $\mathcal{E}(B_z)/(m_\infty^4/g^2)$
- $\mathcal{E}(HL)/(m_\infty^4/g^2)$

$\gamma_* t$

Strong anisotropy $\zeta = 10$ γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)   34

Isotropization - particles

Direction of the momentum surplus

j

F

$\otimes$ $\odot$

B

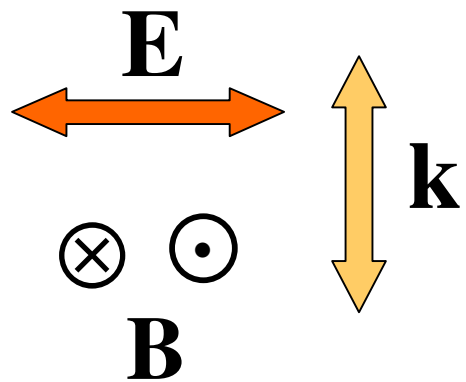$$\Delta \mathbf{p} = \int dt\, \mathbf{F}$$

Isotropization - fields

Direction of the momentum surplus

$$\mathbf{E}$$

$$\mathbf{k}$$

$$\mathbf{B}$$

$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$
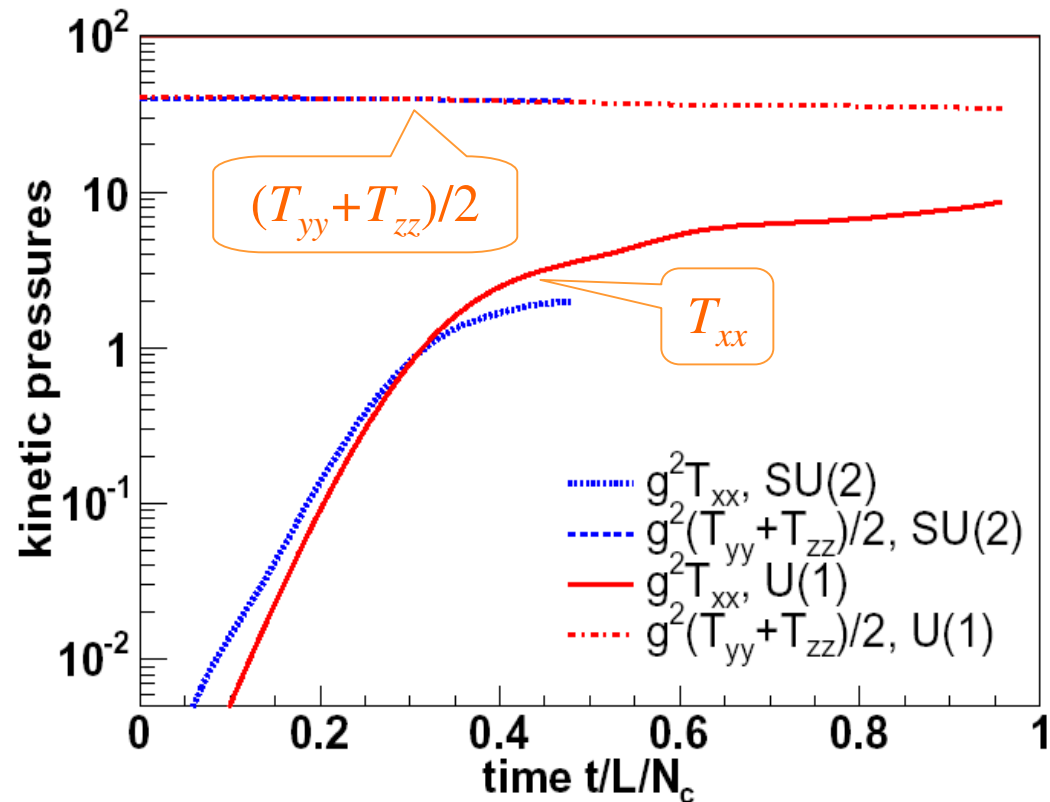
Isotropization – numerical simulation

Classical system of colored particles & fields

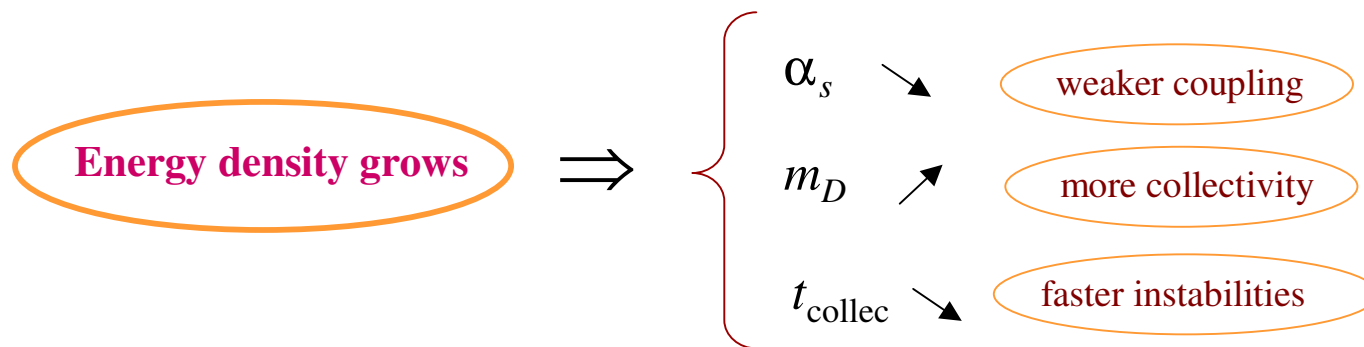$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz})/2$$

A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

37

Instabilities @ LHC

RHIC $\Longrightarrow$ LHC

Energy density grows $\Longrightarrow$

α_s $\searrow$ weaker coupling

m_D $\nearrow$ more collectivity

t_{collec} $\searrow$ faster instabilities

Conclusions

▶ **Instabilities generate chromodynamic fields – QGP is not a gas of partons**

▶ **Instabilities are fast, faster than collisions**

▶ **Instabilities strongly influence plasma dynamics**