# Kinetic theory III

This lecture is devoted to a derivation of hydrodynamics from the kinetic theory. The hydrodynamics is usually used to describe fluids (the term comes from Greek $\upsilon\delta\rho\omega$ – water), however its also applicable to gases. Then, we actually deal with the aerodynamics.

We are going to derive the equations of hydrodynamics starting with the kinetic Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f(t,\mathbf{r},\mathbf{p}) = C(t,\mathbf{r},\mathbf{p}), \tag{1}$$

where, for simplicity, an external force field is absent.

## Macroscopic conservation laws

- As we remember, a structure of collision term $C(t,\mathbf{r},\mathbf{p})$ guarantees that the following relations are satisfied

$$\int \frac{d^3p}{(2\pi)^3}\, C(t,\mathbf{r},\mathbf{p}) = 0, \tag{2}$$

$$\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p}\, C(t,\mathbf{r},\mathbf{p}) = 0, \tag{3}$$

$$\int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}}\, C(t,\mathbf{r},\mathbf{p}) = 0, \tag{4}$$

  because of the particle number, momentum and energy conservations. The particle number conservation holds only for binary collisions but the remaining two laws are general.

- Due to Eqs. (2, 3, 4) we can get macroscopic conservation laws in the form of continuity equations. For this purpose we multiply the Boltzmann equation by $1, p^i$ and $\epsilon_{\mathbf{p}}$, respectively, and perform the integration over momentum. Thus, we find

$$\frac{\partial \rho(t,\mathbf{r})}{\partial t} + \nabla\cdot\mathbf{j}(t,\mathbf{r}) = 0, \tag{5}$$

$$\frac{\partial P^i(t,\mathbf{r})}{\partial t} + \frac{\partial \Pi^{ij}(t,\mathbf{r})}{\partial r^j} = 0, \tag{6}$$

$$\frac{\partial \varepsilon(t,\mathbf{r})}{\partial t} + \nabla\cdot\mathbf{I}(t,\mathbf{r}) = 0, \tag{7}$$

  where $i,j = x,y,z$ and

$$\rho(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, f(t,\mathbf{r},\mathbf{p}), \qquad \mathbf{j}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, \frac{\mathbf{p}}{m}\, f(t,\mathbf{r},\mathbf{p}), \tag{8}$$

$$P^i(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, p^i f(t,\mathbf{r},\mathbf{p}), \qquad \Pi^{ij}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, \frac{p^i p^j}{m}\, f(t,\mathbf{r},\mathbf{p}), \tag{9}$$

$$\varepsilon(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} f(t,\mathbf{r},\mathbf{p}), \qquad \mathbf{I}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, \frac{\mathbf{p}}{m}\, \epsilon_{\mathbf{p}} f(t,\mathbf{r},\mathbf{p}). \tag{10}$$

  The quantities are: $\rho$ – particle density, $\mathbf{j}$ – particle flux, $P^i$ – momentum density, $\Pi^{ij}$ – momentum flux, $\varepsilon$ – energy denisty $\mathbf{I}$ – energy flux.

## Local thermodynamic equilibrium

- Equations of ideal-fluid hydrodynamics are obtained from the macroscopic conservation laws (5, 6, 7) substituting the distribution function of local thermodynamical equilibrium

$$f^{\rm eq}(t,\mathbf{r},\mathbf{p}) = \rho(t,\mathbf{r})\left(\frac{2\pi}{mk_BT(t,\mathbf{r})}\right)^{3/2}\exp\left[-\frac{\big(\mathbf{p}-m\mathbf{u}(t,\mathbf{r})\big)^2}{2mk_BT(t,\mathbf{r})}\right], \tag{11}$$

  into Eqs. (8, 9, 10). We note that $\rho$, $T$ and $\mathbf{u}$ in Eq. (11) depend on $t$ and $\mathbf{r}$ which just makes the function of thermodynamical equilibrium local.

- Actually, behind the formula (11) there is an assumption of fundamental importance that a system, which evolves towards the thermodynamical equilibrium, first approaches the local equilibrium and later on it evolves hydrodynamically to the global equilibrium where the parameters $\rho$, $T$ and $\mathbf{u}$ are time and position independent.

- Substituting the distribution function (11) into Eqs. (8, 9, 10) and taking the integral over momentum, we get

$$\mathbf{j} = \rho\,\mathbf{u}, \tag{12}$$

$$\mathbf{P} = m\rho\,\mathbf{u}, \qquad \Pi^{ij} = m\rho\, u^i u^j + \delta^{ij}\rho\, k_BT, \tag{13}$$

$$\varepsilon = \frac{1}{2}m\rho\,\mathbf{u}^2 + \frac{3}{2}\rho\, k_BT, \qquad \mathbf{I} = \frac{1}{2}m\rho\,\mathbf{u}^3 + \frac{5}{2}\rho\,\mathbf{u}\, k_BT, \tag{14}$$

  where the arguments $t$ and $\mathbf{r}$ of $\rho$, $T$ and $\mathbf{u}$ are suppressed to simplify the notation.

- To take the momentum integrals, which lead to the formulas (12, 13, 14), one introduces the variable $\mathbf{k} \equiv \mathbf{p} - m\mathbf{u}$. The integrals are then split into sums of the integrals with the integrands which are odd or even as a function of $\mathbf{k}$. The integrals with the odd integrands vanish and those with the even ones are computed using the formulas

$$\int_0^\infty dx\, e^{-x^2} = \frac{\sqrt{\pi}}{2}, \qquad \int_0^\infty dx\, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}, \qquad \int_0^\infty dx\, x^4 e^{-x^2} = \frac{3\sqrt{\pi}}{8}. \tag{15}$$

## Hydrodynamics of ideal fluid

- Substituting the flux (12) into the continuity equation (5), we get

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{16}$$

- Using the formula (13) and the equation (16), Eq. (6) becomes the well-known Euler equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} + \frac{1}{m\rho}\nabla p = 0, \tag{17}$$

  where $p$ is the pressure given by the equation of state

$$p = \rho\, k_BT. \tag{18}$$

- Substituting the energy density and energy flux (14) into the continuity equation (4) and using Eqs. (16, 17), one finds

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)T + \frac{2}{3}T\nabla\cdot\mathbf{u} = 0. \tag{19}$$

- The set of five equations (16, 17, 19) constitute the <u>hydrodynamic equations of ideal fluid</u>.
- Since there six unknown functions $\rho, \mathbf{u}, p, T$ one has to include the equation of state (18) to close the system of equations.
- The differential operator

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big), \tag{20}$$

  which is present in Eqs. (17, 19), is called the <u>substantial or material derivative</u>. Acting on a given quantity it gives a temporal change of the quantity in the reference frame which moves with the velocity $\mathbf{u}$.
- The continuity equation (16) written by means of the substantial derivative is

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\rho + \rho\nabla\cdot\mathbf{u} = 0. \tag{21}$$

- The collision term of Boltzmann equation vanishes when computed with the equilibrium distribution function (11). Therefore, as the proof of the $H-$theorem shows, the entropy is then maximal. Consequently, one expects that the temporal evolution driven the hydrodynamic equations of ideal fluid (16, 17, 19) is not associated with the entropy growth that is the evolution is <u>isoentropic</u> or <u>adiabatic</u>.
- The proof of the adiabatic evolution of ideal fluid is the following. It has been discussed in Lecture I that the $TV^\gamma$, where $\gamma \equiv nR/C_V = 2/3$, is conserved in an adiabatic process of an ideal gas. Dividing $TV^{2/3}$ by $N^{2/3}$, where $N$ is the number of particles, we find that the quantity $T\rho^{-2/3}$ is conserved and any power of the quantity, in particular $T^{-3/2}\rho$, is also conserved. Now, we will show that the substantial derivative of $T^{-3/2}\rho$ indeed vanishes if $T$ and $\rho$ satisfy the equations (17, 19, 21).
- We compute

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)T^{-\frac{3}{2}}\rho = -\frac{3}{2}\rho T^{-\frac{5}{2}}\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)T + T^{-\frac{3}{2}}\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\rho = 0, \tag{22}$$

  where the equations (19, 21) have been used. So, we conclude that the hydrodynamic evolution of ideal fluid is isoentropic.
- The hydrodynamic equations (16, 17), we derived in the framework of kinetic theory, are applicable not only for dilute gases but also for liquids, as simple heuristic arguments show. Actually, the Euler equation, which is about hundred years older than the Boltzmann equation, was first obtained for liquids not gases. However, a first-principle derivation of hydrodynamics of liquids remains an unsolved problem.
- The equations (16, 17) are not a closed system of equations because there are five functions: $\rho, \mathbf{u}, p$. When the equations are applied to liquids one often assumes that the liquid is <u>incompressible</u> that $\rho$ is constant. Then, not only the system of hydrodynamic equations becomes closed but the continuity equation (16) tells us that $\nabla\cdot\mathbf{u} = 0$ which means that the field of $\mathbf{u}$ is <u>sourceless</u>. Consequently, an analysis of the Euler equation is simplified.

- The distribution function of local equilibrium (11) does not statisfy the Boltzmann equation. The collision term is then zero but the left-hand-side of the equation is nonzero because $\rho, \mathbf{u}, T$ depend on $t$ and $\mathbf{r}$. To resolve the problem the equilibrium distribution function (11) is replaced by the function

$$f(t, \mathbf{r}, \mathbf{p}) = f^{\mathrm{eq}}(t, \mathbf{r}, \mathbf{p}) + \delta f(t, \mathbf{r}, \mathbf{p}). \tag{23}$$

  Due to the 'small' function $\delta f(t, \mathbf{r}, \mathbf{p})$ the collision term is not exactly zero and the Boltzmann equation can be satisfied.

- Substituting the function (23) into the macroscopic conservation laws we obtain a hydrodynamics of viscous fluid which is no longer isoentropic. This is the subject of the next lecture.