Kinetic theory IV

In this lecture there will be discussed transport phenomena and transport coefficients of heat conductivity and viscosity. The equations of viscous hydrodynamics will be also derived.

Relaxation time approximation

- A complicated structure of the Boltzmann collision term is a serious obstacle in applications of the kinetic theory. The collision term is radically simplified in the relaxation time approximation which is

$$C(t,\mathbf{r},\mathbf{p}) = \frac{1}{\tau}\big(f^{\rm eq}(t,\mathbf{r},\mathbf{p}) - f(t,\mathbf{r},\mathbf{p})\big), \tag{1}$$

  where τ is called the relaxation time discussed later on and $f^{\rm eq}(t,\mathbf{r},\mathbf{p})$ is the distribution function of local equilibrium

$$f^{\rm eq}(t,\mathbf{r},\mathbf{p}) = \rho(t,\mathbf{r})\left(\frac{2\pi}{mk_BT(t,\mathbf{r})}\right)^{3/2}\exp\left[-\frac{\big(\mathbf{p}-m\mathbf{u}(t,\mathbf{r})\big)^2}{2mk_BT(t,\mathbf{r})}\right]. \tag{2}$$

- To clarify a physical meaning of the collision term (1), let us consider a system which is homogeneous. Consequently, the distribution function is position independent and the equilibrium function is time independent. The kinetic equation then equals

$$\frac{\partial f(t,\mathbf{p})}{\partial t} = \frac{f^{\rm eq}(\mathbf{p}) - f(t,\mathbf{p})}{\tau}, \tag{3}$$

  and its solution is

$$f(t,\mathbf{p}) = \big(f(0,\mathbf{p}) - f^{\rm eq}(\mathbf{p})\big)e^{-\frac{t}{\tau}} + f^{\rm eq}(\mathbf{p}). \tag{4}$$

  As seen the system evolves towards equilibrium and $f(t,\mathbf{p}) = f^{\rm eq}(\mathbf{p})$ after the time $t \gg \tau$. The parameter τ is a characteristic time of equilibration.

Rough estimate of τ

To fully specify the collision term (1) the parameter τ needs to be estimated.

- If τ is identified as the mean free flight time of a gas particle it is

$$\tau = \frac{\bar{l}}{\bar{v}}, \tag{5}$$

  where $\bar{v}$ is the average velocity of a gas particle and $\bar{l}$ is its mean free path.

- The velocity $\bar{v}$ is estimated equating the particle kinetic energy $\frac{1}{2}m\bar{v}^2$ to the thermal average energy $\frac{3}{2}k_BT$ which gives

$$\bar{v} = \sqrt{\frac{3k_BT}{m}}. \tag{6}$$

- To get the mean free path $\bar{l}$ we consider a test particle which has just experienced a collision and we ask when the particle will collide again. If the interaction cross section is σ the collision will occur when a gas particle appears in a cylinder of the base area σ and the axis along the test-particle velocity. The cylinder's height equals the mean-free path l and the number of particles in the cylinder equals one. Thus we require $\bar{l}\sigma\rho = 1$ which gives

$$\bar{l} = \frac{1}{\rho\sigma}. \tag{7}$$

- Substituting the formulas (6, 7) into Eq. (5), one gets the following rough estimate of the relaxation time

$$\tau = \frac{1}{\rho\sigma}\sqrt{\frac{m}{3k_BT}}. \tag{8}$$

More accurate estimate of τ

- A more accurate estimate of τ can be obtained comparing the Boltzmann collision term with that of the relaxation time approximation (1). Let us assume that there is the approximate equality

$$\frac{f^{\rm eq}(\mathbf{p}) - f(t,\mathbf{r},\mathbf{p})}{\tau} = \int \frac{d^3p_1}{(2\pi)^3}\, d\Omega\, |\mathbf{v}-\mathbf{v}_1|\, \frac{d\sigma}{d\Omega}\Big[f(t,\mathbf{r},\mathbf{p}')\, f(t,\mathbf{r},\mathbf{p}_1') - f(t,\mathbf{r},\mathbf{p})\, f(t,\mathbf{r},\mathbf{p}_1)\Big]. \tag{9}$$

On both sides of Eq. (9) there are positive and negative terms. So, we require

$$\frac{f(t,\mathbf{r},\mathbf{p})}{\tau} = \int \frac{d^3p_1}{(2\pi)^3}\, d\Omega\, |\mathbf{v}-\mathbf{v}_1|\, \frac{d\sigma}{d\Omega} f(t,\mathbf{r},\mathbf{p})\, f(t,\mathbf{r},\mathbf{p}_1). \tag{10}$$

- Assuming that the cross section weakly depends on the initial momentum $\mathbf{p}+\mathbf{p}_1$ and scattering angle Ω, we can perform the angular integral in Eq. (10) and we get

$$\frac{f(t,\mathbf{r},\mathbf{p})}{\tau} = \frac{\sigma}{m}\int \frac{d^3p_1}{(2\pi)^3}\, |\mathbf{p}-\mathbf{p}_1|\, f(t,\mathbf{r},\mathbf{p})\, f(t,\mathbf{r},\mathbf{p}_1). \tag{11}$$

- If one divides Eq. (11) by $f(t,\mathbf{r},\mathbf{p})$ the relaxation time τ is momentum dependent. This is physically realistic but the relaxation time approximation is no longer simple. We instead take the momentum integral of both sides of Eq. (11) which gives

$$\frac{1}{\tau} = \frac{\sigma}{m\,\rho}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p_1}{(2\pi)^3}\, |\mathbf{p}-\mathbf{p}_1|\, f(t,\mathbf{r},\mathbf{p})\, f(t,\mathbf{r},\mathbf{p}_1). \tag{12}$$

- Now we substitute the local equilibrium distribution (2) with $\mathbf{u}=0$ into Eq. (12). To perform the momentum integrals we introduce the center-of-mass variables: $\mathbf{P} = \frac{1}{2}(\mathbf{p}+\mathbf{p}_1)$ and $\mathbf{q} = \mathbf{p}-\mathbf{p}_1$. The integrals over $\mathbf{P}$ and $\mathbf{q}$ factorize from each other and one gets

$$\frac{1}{\tau} = \frac{\sigma\rho}{m}\Big(\frac{2\pi}{mk_BT}\Big)^3 \int \frac{d^3P}{(2\pi)^3}\exp\Big[-\frac{\mathbf{P}^2}{mk_BT}\Big]\int \frac{d^3q}{(2\pi)^3}|\mathbf{q}|\exp\Big[-\frac{\mathbf{q}^2}{4mk_BT}\Big] = 4\sigma\rho\sqrt{\frac{k_BT}{\pi m}}, \tag{13}$$

which finally gives

$$\tau = \frac{1}{4\rho\sigma}\sqrt{\frac{\pi m}{k_BT}}. \tag{14}$$

- Taking into account that $\frac{1}{\sqrt{3}} \approx 0.58$ and $\frac{\sqrt{\pi}}{4} \approx 0.44$ the estimates (8) and (14) agree with each other surprisingly well. In our further considerations we will ignore the numerical factor and we will use the estimate
$$\tau = \frac{1}{\rho\sigma}\sqrt{\frac{m}{k_B T}}. \tag{15}$$

Nitrogen in normal conditions

To get an idea about magnitudes of parameters, in particular about τ, let us consider nitrogen in the normal conditions that is at the temperature $0\,^\circ\mathrm{C} = 273$ K and the pressure 1 atm = 760 mmHg. Nitrogen, which makes up about 80% of air, is a gas of diatomic molecules N_2 in normal conditions. There are two stable isotopes ^{14}N and ^{15}N but the former one is much more abundant.

- The nitrogen mass density is $1.25 \cdot 10^{-3}\,\mathrm{g\,cm^{-3}}$.
- The atomic mass of ^{14}N is 14 atomic mass units u ($\mathrm{u} = 1.66 \cdot 10^{-24}$ g) that is $2.32 \cdot 10^{-23}$g. The mass of the molecule N_2 equals $m = 4.64 \cdot 10^{-23}$g.
- The density of molecules N_2 is $\rho = 2.69 \cdot 10^{19}\,\mathrm{cm^{-3}}$.
- The diameter of a molecule N_2 is $a \approx 2.5\,\mathrm{\mathring{A}} = 2.5 \cdot 10^{-8}\,\mathrm{cm}$. Since the (classical) cross section of a collision of two balls of a diameter a is $\sigma = \pi a^2$, we get the interaction cross section $\sigma = 2.0 \cdot 10^{-15}\,\mathrm{cm}^2$.
- The mean free path is $\bar{l} = (\rho\sigma)^{-1} = 1.9 \cdot 10^{-5}$ cm.
- At $T = 273$ K the thermal velocity is $\bar{v} = \sqrt{\frac{3k_B T}{m}} = 4.9 \cdot 10^4\ \frac{\mathrm{cm}}{\mathrm{s}}$, where we have used $k_B = 1.38 \cdot 10^{-16}\ \frac{\mathrm{g\,cm^2}}{\mathrm{s^2\,K}}$.
- The relaxation time is finally found as $\tau = \bar{l}\,\bar{v}^{-1} = 3.9 \cdot 10^{-10}$ s.

Solution of kinetic equation

- We look for a solution of the equation
$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big) f(t,\mathbf{r},\mathbf{p}) = \frac{1}{\tau}\big(f^{\mathrm{eq}}(t,\mathbf{r},\mathbf{p}) - f(t,\mathbf{r},\mathbf{p})\big). \tag{16}$$
- Since we expect that a solution of Eq. (16) evolves towards the local thermodynamical equilibrium, we look for the solution of the form
$$f(t,\mathbf{r},\mathbf{p}) = f^{\mathrm{eq}}(t,\mathbf{r},\mathbf{p}) + \delta f(t,\mathbf{r},\mathbf{p}), \tag{17}$$
with
$$f^{\mathrm{eq}}(t,\mathbf{r},\mathbf{p}) \gg |\delta f(t,\mathbf{r},\mathbf{p})|. \tag{18}$$
- Substituting the function (17) into the equation (16) and using the condition (18), we obtain
$$\delta f(t,\mathbf{r},\mathbf{p}) = -\tau D_{\mathbf{v}} f^{\mathrm{eq}}(t,\mathbf{r},\mathbf{p}), \tag{19}$$
where the substantial derivative equals
$$D_{\mathbf{v}} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla. \tag{20}$$

- Taking into account that the equilibrium distribution function depends on t and $\mathbf{r}$ only through $\rho, \mathbf{u}$ and T, we compute the left-hand-side of Eq. (19) as

$$D_{\mathbf{v}} f^{\rm eq} = \frac{\partial f^{\rm eq}}{\partial \rho}\, D_{\mathbf{v}}\rho + \frac{\partial f^{\rm eq}}{\partial u^i}\, D_{\mathbf{v}} u^i + \frac{\partial f^{\rm eq}}{\partial T}\, D_{\mathbf{v}} T, \tag{21}$$

which gives

$$\frac{D_{\mathbf{v}} f^{\rm eq}}{f^{\rm eq}} = \frac{1}{\rho}\, D_{\mathbf{v}}\rho + \frac{p^i - m u^i}{k_B T}\, D_{\mathbf{v}} u^i + \frac{1}{T}\Big(\frac{(\mathbf{p} - m\mathbf{u})^2}{2 m k_B T} - \frac{3}{2}\Big) D_{\mathbf{v}} T. \tag{22}$$

- Since the function $\delta f(t, \mathbf{r}, \mathbf{p})$ is assumed to be small, the functions $\rho, \mathbf{u}$ and T are expected to satisfy the equations of ideal hydrodynamics

$$D_{\mathbf{u}}\rho + \rho \nabla\cdot\mathbf{u} = 0, \tag{23}$$

$$D_{\mathbf{u}}\mathbf{u} + \frac{1}{m\rho}\nabla p = 0, \tag{24}$$

$$D_{\mathbf{u}} T + \frac{2}{3} T \nabla\cdot\mathbf{u} = 0, \tag{25}$$

which allows one to eliminate the time derivative of $\rho, \mathbf{u}$ and T from the right-hand-side of Eq. (22). Additionally expressing the pressure as $p = \rho k_B T$ and using the particle velocity $\mathbf{v} \equiv \frac{\mathbf{p}}{m}$ instead of the momentum $\mathbf{p}$, one obtains

$$\begin{aligned} \frac{D_{\mathbf{v}} f^{\rm eq}}{f^{\rm eq}} &= \frac{m}{k_B T}\Big((v^i - u^i)(v^j - u^j)\nabla^j u^i - \frac{1}{3}(v^i - u^i)(v^i - u^i)\nabla^j u^j\Big) \\ &+ \frac{1}{T}\Big(\frac{m}{2 k_B T}\,(v^i - u^i)(v^i - u^i) - \frac{5}{2}\Big)(v^j - u^j)\nabla^j T. \end{aligned} \tag{26}$$

- Substituting the result (26) into Eq. (19) we finally find

$$\begin{aligned} \delta f &= -\tau f^{\rm eq}\Big[\frac{m}{k_B T}\Big((v^i - u^i)(v^j - u^j)\nabla^j u^i - \frac{1}{3}(v^i - u^i)(v^i - u^i)\nabla^j u^j\Big) \\ &\quad + \frac{1}{T}\Big(\frac{m}{2 k_B T}\,(v^i - u^i)(v^i - u^i) - \frac{5}{2}\Big)(v^j - u^j)\nabla^j T\Big]. \end{aligned} \tag{27}$$

In this way, we have found the approximate solution $f = f^{\rm eq} + \delta f$ of the kinetic equation (16).

Matching conditions

Since the equations of ideal hydrodynamics (23, 24, 25) are used to derive the function (27), the function satisfies the matching conditions to be discussed here.

- The equilibrium distribution function (2) is expressed through ρ, T and $\mathbf{u}$. Consequently,

$$\rho(t, \mathbf{r}) = \int \frac{d^3 p}{(2\pi)^3}\, f^{\rm eq}(t, \mathbf{r}, \mathbf{p}) = \rho(t, \mathbf{r}), \tag{28}$$

$$P^i(t, \mathbf{r}) = \int \frac{d^3 p}{(2\pi)^3}\, p^i f^{\rm eq}(t, \mathbf{r}, \mathbf{p}) = m\rho(t, \mathbf{r})\, u^i(t, \mathbf{r}), \tag{29}$$

$$\varepsilon(t, \mathbf{r}) = \int \frac{d^3 p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} f^{\rm eq}(t, \mathbf{r}, \mathbf{p}) = \frac{1}{2} m\rho(t, \mathbf{r})\, \mathbf{u}^2(t, \mathbf{r}) + \frac{3}{2}\rho\, k_B T(t, \mathbf{r}). \tag{30}$$

- However, the left-hand-sides of the equations (28, 29, 30) should be also reproduced by the function $f(t,\mathbf{r},\mathbf{p})$. Thus, we arrive to the matching conditions:

$$\int \frac{d^3p}{(2\pi)^3}\, f^{\rm eq}(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\, f(t,\mathbf{r},\mathbf{p}), \tag{31}$$

$$\int \frac{d^3p}{(2\pi)^3}\, p^i f^{\rm eq}(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\, p^i f(t,\mathbf{r},\mathbf{p}), \tag{32}$$

$$\int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} f^{\rm eq}(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} f(t,\mathbf{r},\mathbf{p}). \tag{33}$$

which mean that

$$\int \frac{d^3p}{(2\pi)^3}\, \delta f(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\, p^i \delta f(t,\mathbf{r},\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} \delta f(t,\mathbf{r},\mathbf{p}) = 0. \tag{34}$$

- A direct computation shows that the matching conditions (34) are indeed satisfied by the function (27).
- Let us prove the first and the simplest condition (34). For this purpose we compute

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}\, \delta f &= -\tau\rho \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] \\ &\quad \times\Big[\frac{m}{k_BT}\Big(w^iw^j\nabla^j u^i - \frac{1}{3}w^iw^i\nabla^j u^j\Big) + \frac{1}{T}\Big(\frac{m}{2k_BT}\, w^iw^i - \frac{5}{2}\Big) w^j\nabla^j T\Big],\end{aligned} \tag{35}$$

where the local equilibrium distribution function (2) has been used and we have changed the variables $\mathbf{p} \to \mathbf{w} \equiv \mathbf{p}/m - \mathbf{u}$.

- Further on we use the relations

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] w^i = 0, \tag{36}$$

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] w^iw^j = (2\pi)^{3/2}\Big(\frac{k_BT}{m}\Big)^{5/2}\delta^{ij}, \tag{37}$$

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] w^iw^jw^k = 0, \tag{38}$$

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] \mathbf{w}^2 w^iw^j = 5(2\pi)^{3/2}\Big(\frac{k_BT}{m}\Big)^{7/2}\delta^{ij} \tag{39}$$

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] \mathbf{w}^4 w^iw^j = 35(2\pi)^{3/2}\Big(\frac{k_BT}{m}\Big)^{9/2}\delta^{ij}, \tag{40}$$

$$\int d^3w \, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] w^iw^jw^kw^l = (2\pi)^{3/2}\Big(\frac{k_BT}{m}\Big)^{7/2}\big(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\big), \tag{41}$$

which can be easily derived.

- Taking into account the equalities (36, 37, 38), the right-hand-side of Eq. (35) vanishes.
- The remaining two matching conditions (34) are proved in the similar way.
- The matching conditions (34) have been obtained because ρ, T and $\mathbf{u}$ satisfy the equations of ideal hydrodynamics. However, one can invert the reasoning, the matching conditions are required and consequently ρ, T and $\mathbf{u}$ satisfy the equations of ideal hydrodynamics.

Macroscopic quantities

When the ideal hydrodynamics was derived the macroscopic quantities were computed with the distribution function of local equilibrium. Now we are going to discuss what happens when the correction δf given by Eq. (27) is included.

- We are interested in the particle density ρ, particle flux $\mathbf{j}$, momentum density P^i, momentum flux Π^{ij}, energy denisty ε and energy flux $\mathbf{I}$ which are defined as

$$\rho(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, f(t,\mathbf{r},\mathbf{p}), \qquad \mathbf{j}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{m}\, f(t,\mathbf{r},\mathbf{p}), \tag{42}$$

$$P^i(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, p^i f(t,\mathbf{r},\mathbf{p}), \qquad \Pi^{ij}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{p^ip^j}{m}\, f(t,\mathbf{r},\mathbf{p}), \tag{43}$$

$$\varepsilon(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\, \epsilon_{\mathbf{p}} f(t,\mathbf{r},\mathbf{p}), \qquad \mathbf{I}(t,\mathbf{r}) \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{m}\, \epsilon_{\mathbf{p}} f(t,\mathbf{r},\mathbf{p}). \tag{44}$$

- Because of the matching condistions (34), ρ, $\mathbf{j}$, P^i and ε remain as in the ideal hydrodynamics that is

$$\mathbf{j} = \rho\,\mathbf{u}, \qquad \mathbf{P} = m\rho\,\mathbf{u}, \qquad \varepsilon = \frac{1}{2}m\rho\,\mathbf{u}^2 + \frac{3}{2}\rho\, k_BT. \tag{45}$$

- The momentum and energy fluxes Π^{ij} and $\mathbf{I}$ are modified as

$$\Pi^{ij} = m\rho\, u^iu^j + \delta^{ij}\rho\, k_BT + \delta\Pi^{ij}, \tag{46}$$

$$\mathbf{I} = \frac{1}{2}m\rho\,\mathbf{u}^3 + \frac{5}{2}\rho\,\mathbf{u}\, k_BT + \delta\mathbf{I}, \tag{47}$$

  where $\delta\Pi^{ij}$ and $\delta\mathbf{I}$ are the dissipative corrections coming from δf.

- A dissipation is an irreversible energy transfer associated with the entropy growth. A typical example is friction. It will be clear soon why $\delta\Pi^{ij}$ and $\delta\mathbf{I}$ are called dissipative corrections.

Dissipative energy flux

- Let us compute the dissipative energy flux defined as

$$\delta\mathbf{I} \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{m}\,\epsilon_{\mathbf{p}}\delta f. \tag{48}$$

- Using δf given by Eq. (27), one finds

$$\begin{aligned}\delta\mathbf{I} \;&=\; -\frac{\tau\rho\, m^4}{2^{5/2}(\pi m k_B T)^{3/2}}\int d^3w\,(\mathbf{w}+\mathbf{u})^3 \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] \\ &\times\Big[\frac{m}{k_BT}\Big(w^i w^j\nabla^j u^i - \frac{1}{3}\mathbf{w}^2\nabla^j u^j\Big) + \frac{1}{T}\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^j\nabla^j T\Big],\end{aligned} \tag{49}$$

where instead of the momentum $\mathbf{p}$ we use the velocity $\mathbf{w} \equiv \mathbf{p}/m - \mathbf{u}$.

- Ignoring the terms which are odd functions of $\mathbf{w}$ and vanish due to integration over $\mathbf{w}$, the flux (49) is written as

$$\delta\mathbf{I} = \delta\mathbf{I}_1 + \delta\mathbf{I}_2 + \delta\mathbf{I}_3 + \delta\mathbf{I}_4, \tag{50}$$

where

$$\delta\mathbf{I}_1 \;\equiv\; -\frac{\tau\rho\, m^4}{2^{5/2}(\pi m k_B T)^{3/2}T}\int d^3w\,\mathbf{w}^3 \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^j\nabla^j T, \tag{51}$$

$$\delta\mathbf{I}_2 \;\equiv\; -\frac{3\tau\rho\, m^5\mathbf{u}}{2^{5/2}(\pi m)^{3/2}(k_B T)^{5/2}}\int d^3w\,\mathbf{w}^2 \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(w^i w^j\nabla^j u^i - \frac{1}{3}\mathbf{w}^2\nabla^j u^j\Big), \tag{52}$$

$$\delta\mathbf{I}_3 \;\equiv\; -\frac{3\tau\rho\, m^4\mathbf{u}^2}{2^{5/2}(\pi m k_B T)^{3/2}T}\int d^3w\,\mathbf{w} \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^j\nabla^j T, \tag{53}$$

$$\delta\mathbf{I}_4 \;\equiv\; -\frac{\tau\rho\, m^5\mathbf{u}^3}{2^{5/2}(\pi m)^{3/2}(k_B T)^{5/2}}\int d^3w\, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(w^i w^j\nabla^j u^i - \frac{1}{3}\mathbf{w}^2\nabla^j u^j\Big). \tag{54}$$

- Using the formulas (36-40) we find

$$\delta\mathbf{I}_1 = -\frac{5}{2}\tau\rho\, k_B^2 T\,\nabla T, \qquad\qquad \delta\mathbf{I}_2 = \delta\mathbf{I}_3 = \delta\mathbf{I}_4 = 0, \tag{55}$$

which finally gives

$$\delta\mathbf{I} = -\frac{5}{2}\tau\rho\, k_B^2 T\,\nabla T. \tag{56}$$

- It appears that only the temperature gradient contributes to the energy flux while the velocity gradient, which is also present in the expression of δf, does not. A temperature equalization is an irreversible process, which justifies the use of the term dissipative flux.

Heat conductivity

- Substituting the expression (56) into Eq. (47) we find the total energy flux as

$$\mathbf{I} = \frac{1}{2} m\rho\,\mathbf{u}^3 + \frac{5}{2}\rho\,\mathbf{u}\,k_B T - \frac{5}{2}\tau\rho\,k_B^2 T\,\nabla T. \tag{57}$$

The first two terms correspond to the energy transport due to the non-vanishing fluid velocity **u**.

- The heat flow **q** is the energy flux caused by the temperature gradient. So, we write

$$\mathbf{q} = -\frac{5}{2}\tau\rho\,k_B^2 T\,\nabla T. \tag{58}$$

Since the coefficient $\tau\rho\,k_B^2 T$ is positive, the heat flows, as expected, in the direction of temperature decrease.

- The equation (58) agrees with the experimentally established relation known as the Fourier's law of thermal conduction

$$\mathbf{q} = -\kappa\nabla T, \tag{59}$$

where κ is the coefficient of heat conductivity.

- Comparing the relations (58) are (59), one finds κ as

$$\kappa = \frac{5}{2}\tau\rho\,k_B^2 T. \tag{60}$$

- Taking into account the estimate of relaxation time (15), the heat conductivity coefficient is

$$\kappa = \frac{5}{2}\frac{k_B\sqrt{mk_BT}}{\sigma}. \tag{61}$$

The characteristic features of the formula (61) are that κ is independent of gas density and is proportional to the square root of the temperature. Dilute gases indeed manifest such a behavior.

Dissipative momentum flux

- Let us compute the dissipative momentum flux defined as

$$\delta\Pi^{ij} \equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{p^i p^j}{m}\,\delta f. \tag{62}$$

- Substituting δf given by Eq. (27) into the formula (62), one finds

$$\begin{aligned}\delta\Pi^{ij} \;&=\; -\frac{\tau\rho\,m^4}{(2\pi m k_B T)^{3/2}}\int d^3w\,(w^i+u^i)(w^j+u^j)\exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big] \\ &\quad\times\Big[\frac{m}{k_BT}\Big(w^k w^l\nabla^l u^k - \frac{1}{3}\mathbf{w}^2\nabla^k u^k\Big) + \frac{1}{T}\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^k\nabla^k T\Big],\end{aligned} \tag{63}$$

where instead of the momentum **p** we introduced the velocity $\mathbf{w} \equiv \mathbf{p}/m - \mathbf{u}$. Ignoring the terms which are odd functions of **w** and vanish due to integration over **w**, the flux (63) is written as

$$\delta\Pi^{ij} = \delta\Pi_1^{ij} + \delta\Pi_2^{ij} + \delta\Pi_3^{ij} + \delta\Pi_4^{ij}, \tag{64}$$

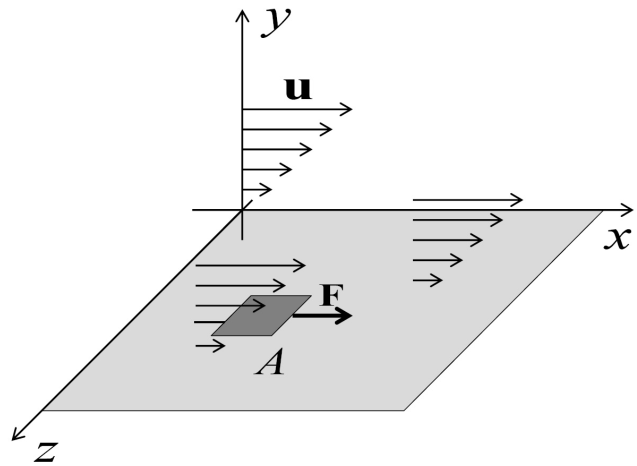

Figure 1: Configuration of fluid velocity used to introduce a viscosity coefficient η

where

$$\delta\Pi_1^{ij} = -\frac{\tau\rho\, m^{7/2}}{(2\pi)^{3/2}(k_BT)^{5/2}}\int d^3w\, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]w^iw^j\Big(w^kw^l\nabla^lu^k - \frac{1}{3}\mathbf{w}^2\nabla^ku^k\Big), \quad (65)$$

$$\delta\Pi_2^{ij} = -\frac{\tau\rho\, m^{5/2}u^j}{(2\pi)^{3/2}(k_BT)^{3/2}T}\int d^3w\, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^iw^k\nabla^kT, \quad (66)$$

$$\delta\Pi_3^{ij} = -\frac{\tau\rho\, m^{5/2}u^i}{(2\pi)^{3/2}(k_BT)^{3/2}T}\int d^3w\, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(\frac{m}{2k_BT}\,\mathbf{w}^2 - \frac{5}{2}\Big)w^jw^k\nabla^kT, \quad (67)$$

$$\delta\Pi_4^{ij} = -\frac{\tau\rho\, m^{7/2}u^iu^j}{(2\pi)^{3/2}(k_BT)^{5/2}}\int d^3w\, \exp\Big[-\frac{m\mathbf{w}^2}{2k_BT}\Big]\Big(w^kw^l\nabla^lu^k - \frac{1}{3}\mathbf{w}^2\nabla^ku^k\Big). \quad (68)$$

- Using the formulas (36-41) we find

$$\delta\Pi_1^{ij} = -\tau\rho\, k_BT\Big[\nabla^iu^j + \nabla^ju^i - \frac{2}{3}\delta^{ij}\nabla^ku^k\Big], \quad (69)$$

$$\delta\Pi_2^{ij} = \delta\Pi_3^{ij} = \delta\Pi_4^{ij} = 0, \quad (70)$$

which finally provides

$$\delta\Pi^{ij} = -\tau\rho\, k_BT\Big[\nabla^iu^j + \nabla^ju^i - \frac{2}{3}\delta^{ij}\nabla^ku^k\Big]. \quad (71)$$

- As seen, the velocity gradient contributes to the momentum flux but the temperature gradient, which is also present in the expression of δf, does not. A velocity equalization is an irreversible process caused by friction. It justifies the use of the term dissipative flux.

- The full momentum flux Π^{ij} is

$$\Pi^{ij} = m\rho\, u^iu^j + \delta^{ij}\rho\, k_BT - \tau\rho\, k_BT\Big[\nabla^iu^j + \nabla^ju^i - \frac{2}{3}\delta^{ij}\nabla^ku^k\Big]. \quad (72)$$

Viscosity

- Let us consider the scheme of gas flow shown in Fig. 1. The fluid velocity $\mathbf{u}$ is along the axis x that is $\mathbf{u} = (u_x, 0, 0)$ with the velocity u_x dependent on y and independent of z.
- Studying experimentally the gas flow in such a configuration, it was found that the friction force F_x per unit area A in the $xz-$plane is proportional to the velocity gradient

$$\frac{F_x}{A} = -\eta \frac{\partial u_x}{\partial y}, \tag{73}$$

  where the proportionality constant η is the viscosity coefficient.

- The component Π^{xy} of the momentum flux tensor equals the momentum along the axis x transported along the axis y per unit time and unit area in the $xz-$plane. In other words, this is the force per unit area in the $xz-$plane acting along the axis x. Therefore, $\Pi^{xy} = F_x/A$.
- Since $\mathbf{u} = (u_x, 0, 0)$, we find from Eq. (72) that

$$\Pi^{xy} = \frac{F_x}{A} = -\tau\rho\, k_B T \frac{\partial u_x}{\partial y} \tag{74}$$

  and comparing it with Eq. (73), we obtain

$$\eta = \tau\rho\, k_B T. \tag{75}$$

- Taking into account the estimate of relaxation time (15), the viscosity coefficient equals

$$\eta = \frac{\sqrt{m k_B T}}{\sigma}. \tag{76}$$

  As seen, it is independent of gas density and proportional to the square root of the temperature.

- Taking the ratio of the coefficients (60) and (75), one finds the relation

$$\frac{\kappa}{k_B \eta} = \frac{5}{2}, \tag{77}$$

  which is supported experimentally.

Hydrodynamics of viscous fluid

- Equations of viscous hydrodynamics are obtained substituting ρ, $\mathbf{j}$, P^i, ε, Π^{ij} and $\mathbf{I}$, which are given as

$$\mathbf{j} = \rho\, \mathbf{u}, \qquad \mathbf{P} = m\rho\, \mathbf{u}, \qquad \varepsilon = \frac{1}{2} m\rho\, \mathbf{u}^2 + \frac{3}{2}\rho\, k_B T, \tag{78}$$

$$\Pi^{ij} = m\rho\, u^i u^j + \delta^{ij}\rho\, k_B T - \eta\Big[\nabla^i u^j + \nabla^j u^i - \frac{2}{3}\delta^{ij}\nabla^k u^k\Big], \tag{79}$$

$$\mathbf{I} = \frac{1}{2} m\rho\, \mathbf{u}^3 + \frac{5}{2}\rho\, \mathbf{u}\, k_B T - \kappa\, \nabla T, \tag{80}$$

into the macroscopic conservation laws

$$\frac{\partial \rho(t,\mathbf{r})}{\partial t} + \nabla\cdot\mathbf{j}(t,\mathbf{r}) = 0, \tag{81}$$

$$\frac{\partial P^i(t,\mathbf{r})}{\partial t} + \nabla^j \Pi^{ij}(t,\mathbf{r}) = 0, \tag{82}$$

$$\frac{\partial \varepsilon(t,\mathbf{r})}{\partial t} + \nabla\cdot\mathbf{I}(t,\mathbf{r}) = 0. \tag{83}$$

- The first equation, which expresses the particle number conservation, is not modified when compared to that of the ideal hydrodynamics because ρ and $\mathbf{j}$ do not have dissipative contributions. So, we have

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\rho + \rho\nabla\cdot\mathbf{u} = 0. \tag{84}$$

- Using Eq. (84), the equation (82) provides the famous Navier–Stokes equation

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)\mathbf{u} + \frac{1}{m\rho}\nabla\Big(p - \frac{\eta}{3}\nabla\cdot\mathbf{u}\Big) - \frac{\eta}{m\rho}\nabla^2\mathbf{u} = 0, \tag{85}$$

where in case under consideration the pressure p is given by the ideal gas equation of state

$$p = \rho\, k_B T. \tag{86}$$

- The Navier–Stokes equation, which becomes the Euler equation when $\eta = 0$, is a pillar of fluid mechanics. Our derivation holds for a dilute gas but the equation is applicable to fluids as well.

- An apparently simple equation (85) is actually very complex and difficult to solve mostly because of its non-linearity. Even in case of incompressible fluid when $\nabla\cdot\mathbf{u} = 0$, general solutions are not known. The equation predicts, in particular, that under certain conditions there occurs a turbulence – chaotic flow of viscous liquid. A secret of the phenomenon, which is still not well understood, is hidden in the Navier–Stokes equation (85).

- Using Eqs. (84, 85), the continuity equation (83) provides the equation

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\Big)T + \frac{2}{3}T\nabla\cdot\mathbf{u} - \frac{2\kappa}{3\rho}\nabla^2 T = 0. \tag{87}$$

- If $\mathbf{u} = 0$, Eq. (87) changes into the equation of heat conductivity

$$\Big(\frac{\partial}{\partial t} - \alpha\nabla^2\Big)T = 0, \tag{88}$$

where $\alpha \equiv \frac{2\kappa}{3\rho}$. Eq. (88) becomes the diffusion equation if T is replaced by the density of diffusing particles and α by the diffusion constant. In contrast to Eq. (87), the linear equation (88) can be rather easily solved.

- Five equations (84, 85, 87) constitute the system of equations of viscous hydrodynamics. Since there are six unknown functions: $\rho, \mathbf{u}, p, T$, the equation of state (86) must be added to close the system.

- An analysis of the equations (84, 85, 87) is at the heart of a vast branch of physics and applied mathematics known as a mechanics of continuous media. The problem is beyond the scope of the lectures. Our goal was to show how viscous hydrodynamics emerges from kinetic theory.