Electromagnetic field

Quantum electrodynamics (QED) is beyond any doubts the most important quantum field theory. This lecture presents the first step towards the formulation of QED – the canonical quantization of non-interacting electromagnetic field. We use the usual in quantum field theory Heaviside system of units where the characteristic factors 4π do no show up in the Maxwell equations but the fine structure constant equals $\alpha \equiv \frac{e^2}{4\pi}$.

Classical electromagnetic field

Lagrange formalism

- The Lagrangian density of electromagnetic field interacting with the four-current $j^\mu \equiv (\rho, \mathbf{j})$ is
$$\mathcal{L}(x) = \frac{1}{4} F^{\mu\nu}(x)\, F_{\nu\mu}(x) - j_\mu(x)\, A^\mu(x), \tag{1}$$
where the strength tensor $F^{\mu\nu}(x)$ is expressed through the four-potential $A^\mu(x) = (\Phi, \mathbf{A})$ as
$$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{2}$$
The covariant form of the Lagrangian (1) clearly shows that it is the Lorentz scalar.

- Since the electric charge is conserved, the current obeys the continuity equation
$$\partial^\mu j_\mu(x) = 0. \tag{3}$$
The current j^μ is treated further on as an external source of the field but not as a dynamical quantity that is it is not influenced by the electromagnetic field.

- In spite of advantages of a covariant notation, a physical meaning is often easier to grasp in the non-covariant notation with the electric and magnetic fields expressed through the potential as
$$\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{4}$$
where the dot denotes the time derivative. The fields are related to the strength tensor as
$$E^i = F^{i0}, \qquad B^i = \frac{1}{2}\epsilon^{ijk} F^{kj}, \tag{5}$$
where ϵ^{ijk} is totally antisymmetric tensor such that $\epsilon^{123} = 1$.

- The non-covariant form of the Lagrangian (1) is
$$\mathcal{L}(x) = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - \rho\Phi + \mathbf{j}\cdot\mathbf{A}. \tag{6}$$

- The strength tensor $F^{\mu\nu}(x)$ transforms under the Lorentz transformation as a tensor
$$F^{\mu\nu}(x) \to F'^{\mu\nu}(x') = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x'), \tag{7}$$
where $\Lambda^\mu{}_\nu$ is the matrix of Lorentz transformation of a four-vector like $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. The potential $A^\mu(x)$ obviously transforms as a four-vector.

- From the definition of the strength tensor (2) it follows that
$$\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0, \tag{8}$$
where the indices are cyclically interchanged in the three terms.

- The equations (8) contain the sourceless Maxwell equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{9}$$

Exercise: Derive Eqs. (9) from Eqs. (8).

- The remaining Maxwell equations are found as the Euler-Lagrange equations

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0, \tag{10}$$

where all components of A^μ are treated as independent variables.

- Using the Lagrangian density (1), Eqs. (10) lead to

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x). \tag{11}$$

- The current continuity equation $\partial^\mu j_\mu = 0$ directly follows for Eq. (11) because $\partial_\nu \partial_\mu F^{\mu\nu} = 0$.
- $\partial_\nu \partial_\mu F^{\mu\nu}$ vanishes as a contraction of symmetric tensor with antisymmetric one.
- Eqs. (11) are equivalent to the Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} = \mathbf{j} + \dot{\mathbf{E}}. \tag{12}$$

Exercise: Derive Eqs. (12) from Eqs. (11).

- Eqs. (11) can be written as

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu, \tag{13}$$

or in the non-covariant way

$$-\nabla^2 \Phi - \nabla \cdot \dot{\mathbf{A}} = \rho, \tag{14}$$

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \nabla \dot{\Phi} + \ddot{\mathbf{A}} = \mathbf{j}. \tag{15}$$

Gauge symmetry

- Equations of electrodynamics (8, 11) are invariant under the gauge transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x), \tag{16}$$

where $\Lambda(x)$ is an arbitrary function. In the non-covariant notation the transformation is

$$\Phi(t,\mathbf{x}) \to \Phi(t,\mathbf{x}) + \dot{\Lambda}(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) \to \mathbf{A}(t,\mathbf{x}) - \nabla\Lambda(t,\mathbf{x}). \tag{17}$$

- The first term of the Lagrangian density (1) is invariant under the transformation (16), even the strength tensor is. The second term of the Lagrangian density (1) seems to be non-invariant as the extra term $j_\mu \partial^\mu \Lambda$ shows up. However, keeping in mind that this is the action $S \equiv \int d^4x \mathcal{L}$ which really matters, we can get rid of the extra term by means of partial integration provided the current is conserved that is $\partial_\mu j^\mu = 0$. So, the action is gauge invariant.

Lorentz gauge

- Due to the gauge invariance of electrodynamics also called the gauge symmetry, we can impose a constraint on the potential which is called the gauge condition or simply the gauge. In this way a problem under study can be simplified.
- One often applies the Lorentz gauge

$$\partial^\mu A_\mu(x) = 0, \tag{18}$$

which is covariant and consequently if imposed in one references it holds in any other frame.
- In the Lorentz gauge the equation (13) gets a simple and elegant form

$$\Box A^\nu(x) = j^\nu(x). \tag{19}$$

Unfortunately, the procedure of quantization is rather complicated in the Lorentz gauge.

Coulomb gauge

- The canonical quantization is usually done in the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0. \tag{20}$$

- A disadvantage of the Coulomb gauge is its non-covariant character. If the condition (20) is imposed in one reference frame, the condition is, in general, not satisfied in other frames.
- In the Coulomb gauge the equations (14, 15) read

$$-\nabla^2 \Phi = \rho, \tag{21}$$

$$-\nabla^2 \mathbf{A} + \nabla\dot{\Phi} + \ddot{\mathbf{A}} = \mathbf{j}. \tag{22}$$

- When charges are absent $\rho = 0$, the Coulomb gauge can be extended to the radiation gauge

$$\Phi = 0, \qquad \nabla \cdot \mathbf{A} = 0. \tag{23}$$

- In the radiation gauge the electric and magnetic fields are

$$\mathbf{E} = -\dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{24}$$

- If not only the charge but also the current vanishes, the electromagnetic field is free and the potential obeys the wave equation

$$\Box \mathbf{A} = 0, \tag{25}$$

which in the physical units is usually written as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0. \tag{26}$$

- A real solution of the wave equation (25) can be written as

$$\mathbf{A}(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \boldsymbol{\epsilon}(\mathbf{k},\lambda)\Big[e^{-ikx}a(\mathbf{k},\lambda) + e^{ikx}a^*(\mathbf{k},\lambda)\Big], \tag{27}$$

where $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ with $\omega_{\mathbf{k}} \equiv |\mathbf{k}|$, $a(\mathbf{k},\lambda)$ is any complex function, and $\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ is the real ($\boldsymbol{\epsilon}(\mathbf{k},\lambda) = \boldsymbol{\epsilon}^*(\mathbf{k},\lambda)$) unit ($\boldsymbol{\epsilon}^2(\mathbf{k},\lambda) = 1$) vector called the polarization vector such that

$$\mathbf{k}\cdot\boldsymbol{\epsilon}(\mathbf{k},\lambda) = 0, \tag{28}$$

which guarantees that the gauge condition $\nabla\cdot\mathbf{A} = 0$ is satisfied. Since in the three-dimensional space there are two linearly independent vectors perpendicular to a given one, there are two polarization vectors labeled by $\lambda = 1, 2$.

- The vectors $\boldsymbol{\epsilon}(\mathbf{k},1)$, $\boldsymbol{\epsilon}(\mathbf{k},2)$ and $|\mathbf{k}|/\mathbf{k}$ form the orthonormal basis in $\mathbb{R}^3$. In particular, it means

$$\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot\boldsymbol{\epsilon}(\mathbf{k},\lambda') = \delta^{\lambda\lambda'}. \tag{29}$$

- The electric and magnetic fields given by the potential (27) are

$$\mathbf{E}(x) = i\sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\,\boldsymbol{\epsilon}(\mathbf{k},\lambda)\Big[e^{-ikx}a(\mathbf{k},\lambda) - e^{ikx}a^*(\mathbf{k},\lambda)\Big], \tag{30}$$

$$\mathbf{B}(x) = i\sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\,\mathbf{k}\times\boldsymbol{\epsilon}(\mathbf{k},\lambda)\Big[e^{-ikx}a(\mathbf{k},\lambda) - e^{ikx}a^*(\mathbf{k},\lambda)\Big], \tag{31}$$

where one recognizes the well-known expressions of the electromagnetic wave with the mutually perpendicular vectors $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{B}$.

Canonical formalism

- Canonical formalism of electromagnetic field faces some difficulties that already arise at the attempt to define the canonical momentum conjugate to A^μ. Following the standard recipe the momentum is defined as

$$\pi_\mu \equiv \frac{\partial\mathcal{L}}{\partial\dot{A}^\mu}. \tag{32}$$

- Neglecting the interaction term in the Lagrangian density (1) and writing it down as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\nu A^\mu, \tag{33}$$

one finds

$$\pi_\mu = \partial_\mu A_0 - \partial_0 A_\mu, \tag{34}$$

which gives

$$\pi_0 = 0, \qquad \pi_i = \partial_i A_0 - \dot{A}_i. \tag{35}$$

Consequently, (A^0, π_0) is not a pair of canonical variables. The problem is not unexpected. Because of the gauge symmetry of electrodynamics the four-potential has non-physical degrees of freedom.

- Using the radiation gauge (23) we get rid of A^0, and we can consider three pairs of variables (A^i, π^i). Since one observes that

$$\boldsymbol{\pi} = \nabla\Phi + \dot{\mathbf{A}} = -\mathbf{E}, \tag{36}$$

the three pairs of canonical variables are $(A^i, -E^i)$.

- The Poisson bracket of the canonical variables is expected to be

$$\{A^i(t,\mathbf{x}), -E^j(t,\mathbf{x}')\}_{\mathrm{PB}} = \delta^{ij}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{37}$$

One observes that the relation is in conflict with both the Gauss law $\nabla\cdot\mathbf{E} = 0$ and the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$. Indeed, when the divergence of the Poisson bracket is taken in respect to $\mathbf{x}$ or $\mathbf{x}'$, the left-hand-side of Eq. (37) vanishes but the left-hand-side does not because $\nabla\delta^{(3)}(\mathbf{x}-\mathbf{x}') \neq 0$ and $\nabla'\delta^{(3)}(\mathbf{x}-\mathbf{x}') \neq 0$. We leave the problem unsolved (a solution requires a modification of the Poisson bracket definition) and look for a Hamiltonian.

- We define the Hamiltonian density in the standard way

$$\mathcal{H} \equiv -\dot{\mathbf{A}}\cdot\mathbf{E} - \mathcal{L} = (\nabla\Phi + \mathbf{E})\cdot\mathbf{E} - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \nabla\Phi\cdot\mathbf{E}. \tag{38}$$

- Going to the Hamiltonian $H = \int d^3x\,\mathcal{H}$, and performing the partial integration the last term drops out. Indeed, one finds

$$\int d^3x\;\nabla\Phi\cdot\mathbf{E} = -\int d^3x\;\Phi\,\nabla\cdot\mathbf{E} = 0, \tag{39}$$

where the Gauss law has been used $\nabla\cdot\mathbf{E} = 0$.

- The final Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2). \tag{40}$$

- One checks using the Maxwell equations that the Hamiltonian density (40) satisfies the continuity equation

$$\dot{\mathcal{H}} + \nabla\cdot\mathbf{P} = 0, \tag{41}$$

where $\mathbf{P} \equiv \mathbf{E}\times\mathbf{B}$ is the Poynting vector.

- The derivation of the Hamiltonian density (40) is uncertain but the energy is conserved as it should.

Exercise: Derive the continuity equation (41).

- Using the expressions of electric and magnetic fields (30, 31), the Hamiltonian equals

$$H = \sum_{\lambda=1}^{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_{\mathbf{k}}\,a^*(\mathbf{k},\lambda)\,a(\mathbf{k},\lambda). \tag{42}$$

Actually, the potential $\mathbf{A}$, which solves the equation (25), has been written as in Eq. (27), to get the Hamiltonian in the form (42).

Exercise: Derive the formula (42).

Quantization of electromagnetic field

Commutation relations

- In a canonically quantized theory fields and conjugate momenta are replaced by operators acting in a space of states. Postulated commutation relations determine how the operators act. Since we have encountered the problem with the Poisson bracket of canonical variables of electromagnetic field, we are going to postulate at first the commutation relations of the annihilation and creation operators and then we will derive the commutation relations satisfied by $\hat{\mathbf{A}}$ and $\hat{\mathbf{E}}$.

- To introduce the annihilation and creation operators we write down the operator counterpart of Eq. (27) as

$$\hat{\mathbf{A}}(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \, \boldsymbol{\epsilon}(\mathbf{k},\lambda) \Big[e^{-ikx} \, \hat{a}(\mathbf{k},\lambda) + e^{ikx} \, \hat{a}^\dagger(\mathbf{k},\lambda) \Big], \tag{43}$$

which satisfies the equation of motion $\Box \hat{\mathbf{A}} = 0$ and the Coulomb gauge condition $\nabla \cdot \hat{\mathbf{A}} = 0$. As in case of other fields the annihilation and creation operators $\hat{a}(\mathbf{k},\lambda)$, $\hat{a}^\dagger(\mathbf{k},\lambda)$ are of the dimension of $m^{-3/2}$.

- We postulate the commutation relations similar to those known from the scalar field

$$[\hat{a}(\mathbf{k},\lambda), \hat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3 \delta^{\lambda\lambda'} \delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{44}$$

$$[\hat{a}(\mathbf{k},\lambda), \hat{a}(\mathbf{k}',\lambda')] = 0, \tag{45}$$

$$[\hat{a}^\dagger(\mathbf{k},\lambda), \hat{a}^\dagger(\mathbf{k}',\lambda')] = 0. \tag{46}$$

- Keeping in mind that

$$\hat{\mathbf{E}}(x) = i \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \, \boldsymbol{\epsilon}(\mathbf{k},\lambda) \Big[e^{-ikx} \, \hat{a}(\mathbf{k},\lambda) - e^{ikx} \, \hat{a}^\dagger(\mathbf{k},\lambda) \Big], \tag{47}$$

and using the completeness relation

$$\sum_{\lambda=1}^{2} \epsilon^i(\mathbf{k},\lambda) \epsilon^j(\mathbf{k},\lambda) = \delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}, \tag{48}$$

one derives the equal-time commutation relations

$$[\hat{A}^i(t,\mathbf{x}), \hat{E}^j(t,\mathbf{x}')] = -i\delta^{ij}_{\perp}(\mathbf{x}-\mathbf{x}'), \tag{49}$$

$$[\hat{A}^i(t,\mathbf{x}), \hat{A}^j(t,\mathbf{x}')] = 0, \tag{50}$$

$$[\hat{E}^i(t,\mathbf{x}), \hat{E}^j(t,\mathbf{x}')] = 0, \tag{51}$$

where $\delta^{ij}_{\perp}$ is the so-called <u>transverse delta</u> defined as

$$\delta^{ij}_{\perp}(\mathbf{x}) \equiv \int \frac{d^3k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{x}} \Big(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \Big). \tag{52}$$

- One checks that the divergence of the transverse delta vanishes that is

$$\nabla^i \delta^{ij}_{\perp}(\mathbf{x}) = \nabla^j \delta^{ij}_{\perp}(\mathbf{x}) = 0. \tag{53}$$

- Because of the property (53) the commutation relation (49) is compatible with the Gauss law $\nabla \cdot \mathbf{E} = 0$ and the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$.

Exercise: Why the relation (48) holds?

Exercise: Derive the relations (49, 50, 51) from Eqs. (44, 45, 46).

- The normally ordered Hamiltonian corresponding to the formula (42) is

$$\hat{H} = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3}\, \omega_{\mathbf{k}}\, \hat{a}^\dagger(\mathbf{k},\lambda)\, \hat{a}(\mathbf{k},\lambda). \tag{54}$$

Space of states and photons

- Since the commutation relations of the annihilation and creation operators and the Hamiltonian (54) are essentially the same as those of the scalar field, the construction of the Fock space is also the same.
- The Hamiltonian is positive definite and the vacuum state $|0\rangle$ exists. The annihilation operators $\hat{a}(\mathbf{k},1)$ and $\hat{a}(\mathbf{k},2)$ annihilate the state that is

$$\hat{a}(\mathbf{k},1)|0\rangle = \hat{a}(\mathbf{k},2)|0\rangle = 0. \tag{55}$$

- The creation operators $\hat{a}^\dagger(\mathbf{k},1)$ and $\hat{a}^\dagger(\mathbf{k},2)$ acting on the vacuum state $|0\rangle$ create the single-particle states – photons – of two possible polarizations. A multiple action of $\hat{a}^\dagger(\mathbf{k},1)$ and $\hat{a}^\dagger(\mathbf{k},2)$ generates the multi-photon states.
- Because of the quantization by means of the commutation relations photons obey the Bose-Einstein statistics.
- Photons are massless as $k^2 = 0$ and their velocity $\mathbf{v} \equiv \frac{\mathbf{k}}{\omega_{\mathbf{k}}}$ equals the speed of light $\mathbf{v}^2 = 1$.
- Since the electromagnetic field is real a photon is a truly neutral particle, it is its own antiparticle.
- A photon is a spin 1 particle but there are only spin projections ± 1 because there is no photon rest frame.

Lorentz covariance

- The quantization procedure is performed in a reference frame where the Coulomb gauge condition is imposed. So, there is no simple way to transform formulas from one frame to another. The Lorentz covariant quantization of electromagnetic field is well-known and understood but the formalism is rather advanced. So, we will present only heuristic arguments how to write the formulas in a covariant form.
- We write down the potential (43) as a four-vector

$$\hat{A}^\mu(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\, \epsilon^\mu(\mathbf{k},\lambda)\Big[e^{-ikx}\, \hat{a}(\mathbf{k},\lambda) + e^{ikx}\, \hat{a}^\dagger(\mathbf{k},\lambda)\Big], \tag{56}$$

introducing the polarization four-vector $\epsilon^\mu(\mathbf{k},\lambda)$.

- In the reference frame where the Coulomb gauge condition is imposed the polarization four-vector is
$$\epsilon^\mu(\mathbf{k},\lambda) \equiv \Big(0, \boldsymbol{\epsilon}(\mathbf{k},\lambda)\Big). \tag{57}$$

- Since the polarization vectors $\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ are perpendicular to $\mathbf{k}$ and mutually perpendicular, see Eqs. (28, 29), there are analogous relations of the four-vectors
$$k^\mu \epsilon_\mu(\mathbf{k},\lambda) = 0, \qquad \epsilon^\mu(\mathbf{k},\lambda)\,\epsilon_\mu(\mathbf{k},\lambda') = -\delta^{\lambda\lambda'}, \tag{58}$$
which have the covariant form.

- We look for the relation analogous to (48) that is we are going to compute
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda). \tag{59}$$

- The problem is greatly simplified if one realizes that the quantity of interest usually appears in the combination
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda)\, j_1^\mu(k)\, j_2^\nu(k), \tag{60}$$
where $j_1^\mu(k)\, j_2^\nu(k)$ are the currents representing single electrons emitting photons. It happens because the electromagnetic interaction is
$$S_I = -\int d^4x\, A^\mu(x) j_\mu(x) = -\int \frac{d^4k}{(2\pi)^4}\, A^\mu(k) j_\mu(k). \tag{61}$$

- Since the current obeys the continuity equation $\partial^\mu j_\mu(x) = 0$, the Fourier transformed current is four-dimensionally transverse $k^\mu j_\mu(k) = 0$.

- So, we look for the expression (60) with the currents which obey $k_\mu j_1^\mu(k) = k_\mu j_2^\mu(k) = 0$.

- Since the expression (60) is a Lorentz scalar it can be computed in any reference frame. We choose that one where the Coulomb gauge condition is imposed. Then, we find
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda)\, j_1^\mu(k)\, j_2^\nu(k) = \sum_{\lambda=1}^{2} \Big(\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot \mathbf{j}_1(k)\Big)\Big(\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot \mathbf{j}_2(k)\Big). \tag{62}$$

- Using the relation (48), we obtain
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda)\, j_1^\mu(k)\, j_2^\nu(k) = \mathbf{j}_1(k)\cdot\mathbf{j}_2(k) - \frac{\Big(\mathbf{k}\cdot\mathbf{j}_1(k)\Big)\Big(\mathbf{k}\cdot\mathbf{j}_2(k)\Big)}{\mathbf{k}^2}. \tag{63}$$

- Since the currents $j_1^\mu \equiv (\rho_1, \mathbf{j}_1)$ and $j_2^\mu \equiv (\rho_2, \mathbf{j}_2)$ are four-dimensionally transverse $\mathbf{k}\cdot\mathbf{j}_1 = |\mathbf{k}|\rho_1$ and $\mathbf{k}\cdot\mathbf{j}_2 = |\mathbf{k}|\rho_2$, the formula (63) can be rewritten as
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda)\, j_1^\mu(k)\, j_2^\nu(k) = \mathbf{j}_1(k)\cdot\mathbf{j}_2(k) - \rho_1(k)\,\rho_2(k) = -j_1^\mu(k)\, j_{2\,\mu}(k). \tag{64}$$

- Finally, we get the desired formula
$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k},\lambda)\,\epsilon_\nu(\mathbf{k},\lambda) = -g_{\mu\nu}, \tag{65}$$
which is frequently used when electromagnetic processes are discussed.