Propagators

When a vacuum expectation value of a product of field operators is computed, there appear objects like $\langle 0|T\,\hat{\phi}(x)\,\hat{\phi}(y)|0\rangle$ called propagators which actually play a very important role in quantum field theory. Our aim is to introduce the propagators and discuss their properties. However, our discussion is limited to free fields.

Propagator of scalar real field

Field-theory definition

- The Feynman propagator is defined as

$$i\Delta(x,y) \equiv \langle 0|T\,\hat{\phi}(x)\,\hat{\phi}(y)|0\rangle, \tag{1}$$

where T denotes the chronological ordering which in case of bosonic operators act as

$$T\hat{\phi}(x)\,\hat{\phi}(y) \equiv \theta(x_0-y_0)\,\hat{\phi}(x)\,\hat{\phi}(y) + \theta(y_0-x_0)\,\hat{\phi}(y)\,\hat{\phi}(x). \tag{2}$$

- The free field propagator (1) can be computed directly from the definition. Using the field decomposition into plane waves

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \left[e^{-ikx}\hat{a}(\mathbf{k}) + e^{ikx}\hat{a}^\dagger(\mathbf{k})\right], \tag{3}$$

the commutation relations

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{4}$$

$$[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0, \tag{5}$$

and the normalization condition $\langle 0|0\rangle = 1$, one immediately finds

$$i\Delta(x,y) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \Big[e^{-ik(x-y)}\theta(x_0-y_0) + e^{ik(x-y)}\theta(y_0-x_0)\Big], \tag{6}$$

where $\omega_{\mathbf{k}} \equiv \sqrt{m^2+\mathbf{k}^2}$ and $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$.

- The formula (6) shows that the Feynman propagator is a sum of two types of plane waves. The waves with positive energy propagate forward in time and those with negative energy backward in time.

- There is a following physical picture behind the formula (6). At the moment of time y_0, which is earlier than x_0, that is $y_0 < x_0$, there is generated from vacuum the particle at $\mathbf{y}$ and it travels to $\mathbf{x}$ where it is annihilated at x_0. If $x_0 < y_0$, an antiparticle is generated at $\mathbf{y}$, it travels backward in time to $\mathbf{x}$ and it is annihilated at x_0.

- The formula (6) shows that the propagator does not depend on x and y but only on $x-y$. It reflects the translational invariance of vacuum. Consequently, the propagator can be written as

$$i\Delta(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \Big[e^{-ikx}\theta(x_0) + e^{ikx}\theta(-x_0)\Big]. \tag{7}$$

- The propagator (7) can be expressed in an elegant form as

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ikx}}{k^2-m^2+i0^+}, \tag{8}$$

where $k^\mu = (k^0, \mathbf{k})$.

- The infinitesimal element $i0^+$ determines how the poles of the integrant $k^0 = \pm\omega_{\mathbf{k}}$ should be circumvented. Either the integral is taken along the green line from Fig. 1 or equivalently, one solves the equation $k^2 - m^2 + i0^+ = 0$ taking into account the infinitesimal imaginary part. Then, $k^0 = \pm\omega_{\mathbf{k}} \mp i0^+$ and the poles are shifted from the axis of real k_0.

- We note that in the formula (7) the four-momentum k is on mass-shell $k^2 = m^2$ which means $k^0 = \pm\omega_{\mathbf{k}}$. In the formula (8) the components of the four-momentum k are all independent from each other.

Exercise: Prove the equivalence of the formulas (7) and (8), performing the integration over k^0 in the integral (8) by means of the Cauchy formula. Distinguish the case $x_0 > 0$ from $x_0 < 0$.

Exercise: Prove that the propagator (1) satisfies the equation

$$[\partial_\mu \partial^\mu + m^2]\Delta(x) = -\delta^{(4)}(x).$$

Use the field commutation relations and take into account that $\frac{d}{dt}\theta(t) = \delta(t)$.

Green's function of Klein-Gordon equation

- One often uses the so-called Green's function to solve inhomogeneous differential equations. Let us consider the Klein-Gordon equation with a source

$$[\partial_\mu \partial^\mu + m^2]\,\phi(x) = j(x), \tag{9}$$

which describes the classical scalar field $\phi(x)$ coupled to the source $j(x)$. The source is external that is the field ϕ does not change it.

- The general solution of the inhomogeneous equation (9) can be written as

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x-x')\, j(x'), \tag{10}$$

where $\phi_0(x)$ is the general solution of the homogeneous equation

$$[\partial_\mu \partial^\mu + m^2]\,\phi_0(x) = 0, \tag{11}$$

which is a superposition of plane waves, and $G(x)$ is the Green's function which obeys the equation

$$[\partial_\mu \partial^\mu + m^2]G(x) = -\delta^{(4)}(x). \tag{12}$$

One easily checks that the formula (10) indeed solves Eq. (9).

- To get an explicit form of the solution (10) we need an explicit form of the Green's function. One solves Eq. (12) by means of the Fourier transform

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} G(k), \tag{13}$$

which substituted in Eq. (12) provides the algebraic equation

$$[k^2 - m^2]G(k) = 1, \tag{14}$$

immediately solved by

$$G(k) = \frac{1}{k^2 - m^2}. \tag{15}$$

- The Green's function thus equals

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ikx}}{k^2 - m^2}, \tag{16}$$

but the expression (16) is not well defined as the integrand is singular − the denominator vanishes at $k^2 = m^2$.

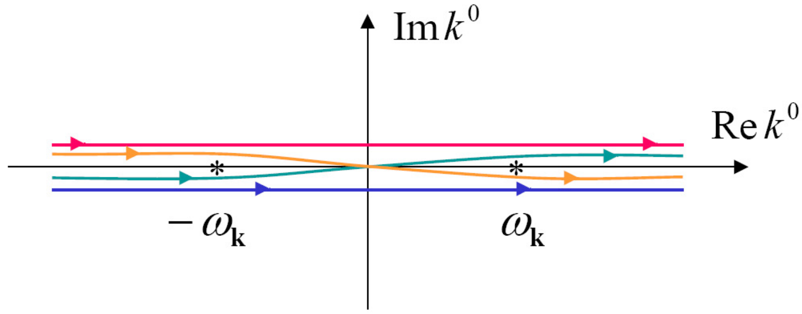

Figure 1: Four possible ways to circumvent the poles of the integrand (16) which correspond to four Green's functions: 'retarded' (red line), 'advanced' (blue line), 'Feynman' (green line) and 'antiFeynman' (orange line)

- To resolve the ambiguity we write down the formula (16) as

$$G(x) = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{e^{-ik_0x_0+i\mathbf{k}\cdot\mathbf{x}}}{k_0^2 - \omega_{\mathbf{k}}^2} = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{e^{-ik_0x_0+i\mathbf{k}\cdot\mathbf{x}}}{(k_0 - \omega_{\mathbf{k}})(k_0 + \omega_{\mathbf{k}})}. \tag{17}$$

  As seen the integrand has two poles at $k_0 = \omega_{\mathbf{k}}$ and $k_0 = -\omega_{\mathbf{k}}$. When one takes the integral over k_0 from $-\infty$ to ∞ there are four ways to circumvent the poles which are shown in Fig. 1. The integral is taken by means of the Cauchy formula and it should be realized that the contour can be closed with the big upper semicircle for $x_0 > 0$ and with the big lower semicircle for $x_0 < 0$. Then, the integrand vanishes along the big semicircle.

- Each way to circumvent the poles of the integrand (16) corresponds to a different Green's function.
  - If the contour runs above both poles (red line), we get the 'retarded' Green's $G_R(x)$ which vanishes for $x_0 < 0$.
  - If the contour runs below both poles (blue line), we get the 'advanced' Green's $G_A(x)$ which vanishes for $x_0 > 0$.
  - If the contour runs below the pole $k_0 = -\omega_{\mathbf{k}}$ and above the pole $k_0 = \omega_{\mathbf{k}}$ (green line), we get the 'Feynman' Green's $G_F(x)$ where the positive k_0 contribute for $x_0 > 0$ and negative k_0 for $x_0 < 0$.
  - If the contour runs above the pole $k_0 = -\omega_{\mathbf{k}}$ and below the pole $k_0 = \omega_{\mathbf{k}}$ (orange line), we get the 'antiFeynman' Green's $G_{\bar{F}}(x)$ where the positive k_0 contribute for $x_0 < 0$ and negative k_0 for $x_0 > 0$.

- The functions can be written as

$$G_R(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i\,\mathrm{sgn}(k_0)0^+}, \tag{18}$$

$$G_A(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 - i\,\mathrm{sgn}(k_0)0^+}, \tag{19}$$

$$G_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i0^+}, \tag{20}$$

$$G_{\bar{F}}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 - i0^+}, \tag{21}$$

  where $\mathrm{sgn}(k_0)$ the sign function such that

$$\mathrm{sgn}(k_0) \equiv \begin{cases} -1 & \text{for} \quad k_0 < 0, \\ +1 & \text{for} \quad k_0 > 0. \end{cases} \tag{22}$$

- We see that the propagator (1) is just the Feynman Green's function of the Klein-Gordon equation.

Propagator of spinor field

- The propagator of the spinor field is defined as

$$iS_{\alpha\beta}(x,y) \equiv \langle 0|T\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y)|0\rangle, \tag{23}$$

  where the operator of chronological ordering T acts on fermionic field operators as

$$T\hat{\psi}(x)\,\hat{\bar{\psi}}(y) \equiv \theta(x_0-y_0)\,\hat{\psi}(x)\,\hat{\bar{\psi}}(y) - \theta(y_0-x_0)\,\hat{\bar{\psi}}(y)\,\hat{\psi}(x). \tag{24}$$

  The sign minus occurs due to the anticommuting character of the spinor field.

- The propagator S can be computed directly from the definition (23), as in case of scalar field. However, we can first derive the equation satisfied by the propagator. Keeping in mind that the fields $\hat{\psi}$, $\hat{\bar{\psi}}$ obey the the Dirac equations

$$[i\gamma_\mu\partial^\mu - m]\hat{\psi}(x) = 0, \qquad \hat{\bar{\psi}}(x)[i\gamma_\mu \overleftarrow{\partial^\mu} + m] = 0, \tag{25}$$

  and using the anticummutation relations

$$\{\hat{\psi}_\alpha(t,\mathbf{x}), \hat{\pi}_\beta(t,\mathbf{x}')\} = i\delta_{\alpha\beta}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{26}$$

  where $\hat{\pi}(x) = i\hat{\psi}^\dagger(x)$, one finds

$$[i\gamma_\mu\partial^\mu_x - m]S(x,y) = \delta^{(4)}(x-y), \qquad S(x,y)[i\gamma_\mu \overleftarrow{\partial^\mu_y} + m] = \delta^{(4)}(x-y). \tag{27}$$

<u>Exercise</u>: Derive the equations (27) starting with the definition (23).

- Eqs. (27) clearly show that the propagator $S(x,y)$, as the scalar-field propagator, depends on x and y only through the difference $x-y$ which reflects the translational invariance of vacuum. So, we can write $S(x-y)$ or $S(x)$.

- We solve Eqs. (27) using the Fourier transform

$$S(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ipx} S(p), \tag{28}$$

  which changes the equation (27) into

$$[\gamma_\mu p^\mu - m]S(p) = 1. \tag{29}$$

- Keeping in mind that $(\gamma_\mu p^\mu - m)(\gamma_\nu p^\nu + m) = p^2 - m^2$, one easily finds

$$S(p) = \frac{\gamma_\mu p^\mu + m}{p^2 - m^2}. \tag{30}$$

- The Feynman propagator of spinor field finally equals

$$S(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{\gamma_\mu p^\mu + m}{p^2 - m^2 + i0^+}. \tag{31}$$

- We note that the spinor-field propagator is related to that of scalar field as

$$S(x) = [i\gamma_\mu\partial^\mu + m]\Delta(x). \tag{32}$$

Propagator of electromagnetic field

- We are interested in the propagator

$$iD^{\mu\nu}(x,y) \equiv \langle 0|T\hat{A}^\mu(x)\hat{A}^\nu(y)|0\rangle. \tag{33}$$

Since the field $\hat{A}^\mu$ is bosonic the operator of chronological ordering T acts as in case of scalar field (2).

- The form of the propagator of electromagnetic field strongly depends on a gauge condition which is chosen. Since we have used the radiation gauge $\hat{A}^0(x) = 0,\ \nabla \cdot \hat{\mathbf{A}}(x) = 0$, we see that $D^{\mu\nu}(x,y) = 0$ if $\mu = 0$ or $\nu = 0$.

- Let us compute D^{ij} directly from the definition (33). Using the free field decomposed into plane waves that is

$$\hat{\mathbf{A}}(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\, \boldsymbol{\epsilon}(\mathbf{k},\lambda)\big[e^{-ikx}\,\hat{a}(\mathbf{k},\lambda) + e^{ikx}\,\hat{a}^\dagger(\mathbf{k},\lambda)\big], \tag{34}$$

and the commutation relations

$$[\hat{a}(\mathbf{k},\lambda), \hat{a}^\dagger(\mathbf{k}',\lambda')] \;=\; (2\pi)^3\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{35}$$

$$[\hat{a}(\mathbf{k},\lambda), \hat{a}(\mathbf{k}',\lambda')] \;=\; [\hat{a}^\dagger(\mathbf{k},\lambda), \hat{a}^\dagger(\mathbf{k}',\lambda')], \tag{36}$$

one obtains

$$iD^{ij}(x,y) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda=1}^{2} \epsilon^i(\mathbf{k},\lambda)\,\epsilon^j(\mathbf{k},\lambda)\Big[e^{-ik(x-y)}\theta(x_0-y_0) + e^{ik(x-y)}\theta(y_0-x_0)\Big], \tag{37}$$

where $\omega_{\mathbf{k}} \equiv |\mathbf{k}|$ and $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$. As previously the propagator depends on x and y only through $x-y$. So, setting $y=0$ we write $D^{ij}(x)$.

Exercise: Derive the formula (37) from the definition (33).

- Using the relation

$$\sum_{\lambda=1}^{2} \epsilon^i(\mathbf{k},\lambda)\epsilon^j(\mathbf{k},\lambda) = \delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}, \tag{38}$$

the propagator (37) can be expressed as

$$iD^{ij}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}\Big(\delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}\Big)\Big[e^{-ikx}\theta(x_0) + e^{ikx}\theta(-x_0)\Big]. \tag{39}$$

- As in case of scalar and spinor fields, the photon propagator is finally written as

$$D^{ij}(x) = \int \frac{d^4k}{(2\pi)^4}\Big(\delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}\Big)\frac{e^{-ikx}}{k^2+i0^+}. \tag{40}$$

And again, the components of k^μ are all independent from each other in Eq. (40).

- The propagator in momentum space is

$$D^{ij}(k) = \Big(\delta^{ij} - \frac{k^ik^j}{\mathbf{k}^2}\Big)\frac{1}{k^2+i0^+}. \tag{41}$$

- The expression (40) or (41) is not very useful. Since it is not covariant, it holds only in the reference frame where the gauge condition is imposed.

- There is also a more serious problem. In the absence of charges one can put $A^0(x) = 0$ but in general it is not possible. Therefore, the expression (40) does not carry complete information about the photon propagator.

- To get the complete photon propagator in a covariant form we refer to a heuristic reasoning. However, our result can be also derived in a rigorous way.

- We assume that $D^{\mu\nu}(k)$ depends only on k^μ. As a Lorentz tensor of the second rank it can be written as

$$D^{\mu\nu}(k) = g^{\mu\nu} f + k^\mu k^\nu f_L, \tag{42}$$

  where f and f_L are unknown functions which are Lorentz scalars.

- We require that the spatial part of (42) agrees with that of (41) which gives

$$-\delta^{ij} f + k^i k^j f_L = \Big(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\Big)\frac{1}{k^2}, \tag{43}$$

  where temporarily the infinitesimal imaginary element $i0^+$ has been ignored for simplicity.

- Taking the trace of Eq. (43) and multiplying it by $k^i k^j$ we get two equations

$$-3f + \mathbf{k}^2 f_L = \frac{2}{k^2}, \qquad -f + \mathbf{k}^2 f_L = 0, \tag{44}$$

  which give

$$f = -\frac{1}{k^2}, \qquad f_L = -\frac{1}{\mathbf{k}^2 k^2}. \tag{45}$$

- We are going to show that the longitudinal part of the propagator f_L is not physical and can be arbitrary chosen.

- Since the electromagnetic interaction is $A^\mu(x)\, j_\mu(x)$, the photon propagator enters scattering amplitudes in the combination $j_\mu(k)\, D^{\mu\nu}(k)\, j_\nu(k)$. Because of charge conservation $\partial^\mu j_\mu(x) = 0$ and $k^\mu j_\mu(k) = 0$, one finds

$$j_\mu(k)\, D^{\mu\nu}(k)\, j_\nu(k) = j_\mu(k)\, j^\mu(k)\, f. \tag{46}$$

  So, the longitudinal contribution f_L has disappeared.

- In quantum theory a gauge freedom is a freedom to choose the longitudinal part of photon propagator.

- The most common is the Feynman gauge with $f_L = 0$. Then, the photon propagator is

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 + i0^+}. \tag{47}$$

- We note that the Fourier transform of (47) satisfies the equation

$$\Box D^{\mu\nu}(x) = g^{\mu\nu}\delta^{(4)}(x). \tag{48}$$

  So, it is the Green's function of the classical potential equation in the Lorentz gauge $\partial^\mu A_\mu(x) = 0$.

- The propagator, which has properties similar to that of classical potential in the Lorentz gauge, is the Landau propagator

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 + i0^+}, \tag{49}$$

  where $f_L = (k^2)^{-2}$. The propagator is purely transverse, that is $k_\mu D^{\mu\nu}(k) = k_\nu D^{\mu\nu}(k) = 0$, as the classical potential in the Lorentz gauge $k_\mu A^\mu(k) = 0$. Sometimes it greatly simplifies calculations.