Quantum Chromodynamics

The quantum chromodynamics (QCD), which is a theory with the nonAbelian SU(3) gauge symmetry, describes interactions of quarks and gluons.

Formulation of QCD

- There are gluons described by a vector field $A^\mu(x)$ and quarks of 6 flavors u, d, s, c, b, t represented by a set of quark fields $\psi_f(x)$.
- The gluon field $A^\mu(x)$ is the 3×3 matrix in the fundamental representation of the SU(3) group which can be expressed as $A^\mu(x) = A^\mu_a(x)\,\tau^a$ where τ^a with $a = 1, 2, \ldots 8$ are generators of the SU(3) group.
- The generators obey the commutation relations

$$[\tau^a, \tau^b] = i f^{abc} \tau^c, \tag{1}$$

where f^{abc} are totally antisymmetric structure constants of SU(3) group. The generators are hermitian traceless matrices normalized in the canonical way as

$$\mathrm{Tr}[\tau^a \tau^b] = \frac{1}{2}\delta^{ab}. \tag{2}$$

- The quark fields are the Dirac spinors ψ_f where the index f numerates quark flavors u, d, s, c, b, t. The quark field carries the spinor index $\alpha = 1, 2, 3, 4$ and the color index $i = 1, 2, 3$.
- The QCD Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[F^{\mu\nu} F_{\nu\mu}] + \sum_f \overline{\psi}_f \left(i D^\mu \gamma_\mu - m_f \right) \psi_f. \tag{3}$$

where the strength tensor is

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu] = D^\mu A^\nu - D^\nu A^\mu = \frac{i}{g}[D^\mu, D^\nu], \tag{4}$$

with $D^\mu \equiv \partial^\mu \mathbb{1} - igA^\mu$ being the covariant derivative.

- The first term of the Lagrangian density (3) can be written as

$$\frac{1}{2}\mathrm{Tr}[F^{\mu\nu} F_{\nu\mu}] = \frac{1}{4} F^{\mu\nu}_a F_{a\,\nu\mu}. \tag{5}$$

- The Lagrangian density (3) is invariant under the gauge transformation

$$\psi_f(x) \quad \rightarrow \quad U(x)\psi_f(x), \tag{6}$$

$$A^\mu(x) \quad \rightarrow \quad U(x)\, A^\mu(x)\, U^\dagger(x) - \frac{i}{g}\big(\partial^\mu U(x)\big) U^\dagger(x), \tag{7}$$

where $U(x)$ is a <u>local</u> SU(3) matrix

$$U(x) = e^{i\omega^a(x)\,\tau^a}, \tag{8}$$

with the x-dependent parameters $\omega^a(x)$.

- The equations of motion are

$$\big[i\gamma_\mu D^\mu - m \big] \psi_f(x) = 0, \tag{9}$$

$$[D_\mu, F^{\mu\nu}(x)] = j^\nu(x), \tag{10}$$

where $j^\mu = j^\mu_a \tau_a$ with $j^\mu_a \equiv g \sum_f \overline{\psi}_f \gamma^\nu \tau_a \psi_f$.

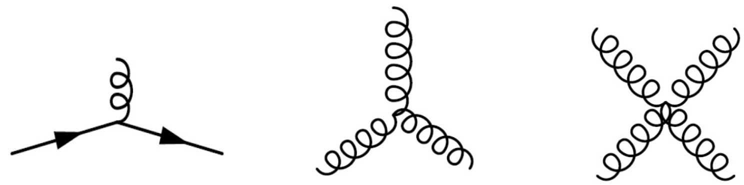

Figure 1: Quark-gluon (left), three-gluon (middle) and four-gluon (right) couplings

- The equations (10), which are called the Yang-Mills equations, can be rewritten as

$$\mathcal{D}_\mu^{ab} F_b^{\mu\nu} = j_a^\nu, \tag{11}$$

where

$$\mathcal{D}_{ab}^\mu = \partial^\mu \delta^{ab} - g f^{abc} A_c^\mu, \tag{12}$$

and

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu. \tag{13}$$

- From the equation (10), one immediately finds that

$$[D_\mu, j^\mu] = 0, \tag{14}$$

that is the quark current is not conserved but covariantly conserved.

Perturbative QCD

- If the coupling $\alpha_s = \frac{g^2}{4\pi}$ is much smaller than unity we deal with the perturabtive QCD.
- Scattering process of quarks and gluons can be described in terms of of Feynman diagrams.
- There three types of couplings shown in Fig. 1 where the solid line denotes a quark and the wavy line a gluon.
- As an example, the three diagrams representing an amplitude of the gluon-quark scattering – analogue of the Compton scattering – are shown in Fig. 2. The first two diagrams are as in QED but the third one is specific for QCD.

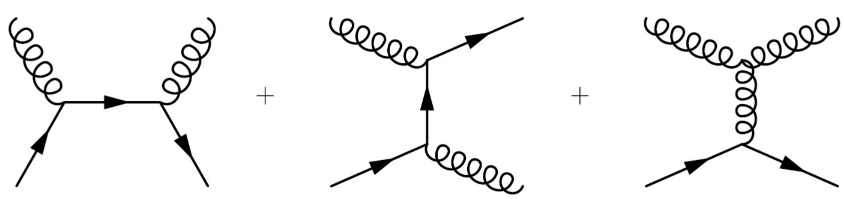

Figure 2: Gluon-quark scattering

Running coupling constant in QED

- One computes a first order correction to the free photon propagator computing the one-loop photon self-energy $\Pi^{\mu\nu}(k)$ shown in Fig. 3.
- Since $\Pi^{\mu\nu}(k)$ is ultraviolet divergent it requires a regularization.

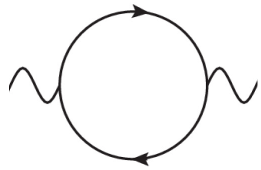

Figure 3: Photon self-energy

- Because of the gauge invariance $k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0$. Consequently, the Lorentz structure of $\Pi^{\mu\nu}(k)$ is
$$\Pi^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\, P(k^2). \tag{15}$$
- The resumed photon propagator in the Feynman gauge, which is shown in Fig. 4, is
$$D^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2\big(1 - P(k^2)\big)}. \tag{16}$$
- The renormalized propagator $D^{\mu\nu}(k,\mu)$ is obtained as
$$D^{\mu\nu}(k,\mu) = \frac{1}{Z_3(\mu)} D^{\mu\nu}(k), \tag{17}$$
where $Z_3(\mu)$ is the renormalization constant and μ is the renormalization scale. When $k^2 \to -\mu^2$ the renormalized propagator coincides with the free one which is the renormalization condition.
- One finds that $Z_3(\mu) = 1 + P(-\mu^2)$ and consequently
$$P(k^2,\mu^2) = P(k^2) - P(-\mu^2), \qquad D^{\mu\nu}(k,\mu) = \frac{g^{\mu\nu}}{k^2\big(1 - P(k^2,\mu^2)\big)}. \tag{18}$$
- Since physical results must be independent of μ, the coupling constant needs to be renormalized.
- Due to the gauge invariance of QED, the coupling constant is renormalized with the same constant $Z_3(\mu)$ as the photon propagator that is
$$\alpha(\mu) = Z_3(\mu)\alpha. \tag{19}$$
- Since the bare coupling α is independent of μ, the renormalized $\alpha(\mu)$ satisfies the equation
$$\mu \frac{d\alpha(\mu)}{d\mu} = \beta(\mu), \tag{20}$$
where $\beta(\mu)$ is the beta function defined as
$$\beta(\mu) = \mu \frac{dZ_3(\mu)}{d\mu} \frac{\alpha(\mu)}{Z_3(\mu)}. \tag{21}$$
- Knowing the explicit expression of $P(\mu^2)$, one finds that
$$\beta(\mu) = \frac{2}{3\pi}\, \alpha^2(\mu). \tag{22}$$

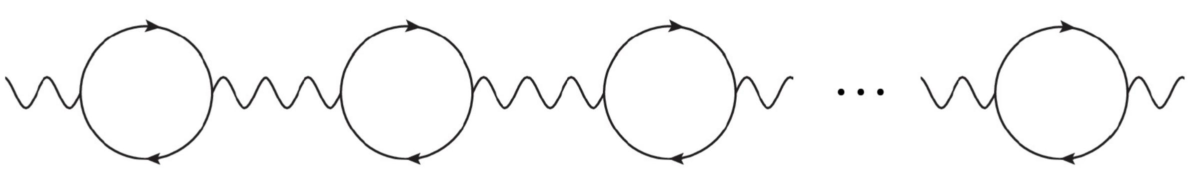

Figure 4: Resumed photon propagator

- One easily checks that the solution of Eq. (20) is

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{\alpha(\mu_0)}{3\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right)}. \tag{23}$$

- Let us define the energy scale Λ such that

$$1 - \frac{\alpha(\mu_0)}{3\pi} \ln\left(\frac{\Lambda^2}{\mu_0^2}\right) = 0. \tag{24}$$

  When $\mu = \Lambda$ the coupling constant $\alpha(\Lambda) = \infty$. The is so-called Landau pole.

- Solving Eq. (24) with respect of $\alpha(\mu_0)$, one finds

$$\alpha(\mu) = \frac{3\pi}{\ln\left(\frac{\Lambda^2}{\mu^2}\right)}, \tag{25}$$

  where μ_0 is replaced by μ and it is assumed that $\mu < \Lambda$.

- The equation (25) expresses the dimensionless coupling constant $\alpha(\mu)$ through the dimensionfull parameter Λ. This is the phenomenon of dimensional transmutation.

- Using the scale $\mu_0 = m_e$ and $\alpha(\mu_0) = 1/137$, one finds that $\Lambda \approx 10^{281} m_e \approx 10^{287}$ eV.

Asymptotic freedom

- In QCD the one-loop beta function is

$$\beta(\mu) = -(33 - 2N_f)\frac{\alpha_s(\mu)}{12\pi}, \tag{26}$$

  where N_f is the number of light flavors of masses much smaller than μ. For $N_f < 17$ the beta functions is negative.

- The running coupling constant evolves from μ_0 to μ according to the formula

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + (33 - 2N_f)\frac{\alpha_s(\mu_0)}{12\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right)}. \tag{27}$$

- The QCD scale parameter Λ_{QCD} is defined through the equation

$$1 + (33 - 2N_f)\frac{\alpha_s(\mu_0)}{12\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right) = 0. \tag{28}$$

  At $\mu = \Lambda_{\text{QCD}}$ the coupling $\alpha_s(\mu)$ becomes infinite.

- Solving Eq. (28) with respect of $\alpha_s(\mu_0)$, one finds

$$\alpha_s(\mu) = \frac{12\pi}{(33 - 2N_f) \ln\left(\frac{\mu^2}{\Lambda_{\text{QCD}}^2}\right)}, \tag{29}$$

  where μ_0 is replaced by μ and it is assumed that $\mu < \Lambda_{\text{QCD}}$.

- If $\mu \to \infty$, the coupling constant $\alpha_s(\mu)$ tends to zero which is known as the asymptotic freedom.

- Experiment shows that $\Lambda_{\text{QCD}} \approx 200$ MeV.

- The renormalization scale is usually identified with a characteristic momentum transfer Q of a process of interest.

- When $Q^2 \gg \Lambda_{\text{QCD}}^2$ a process is hard and can be descried in terms of perturbative QCD.

- A summary of measurements of the QCD running constant is shown in Fig. 5.

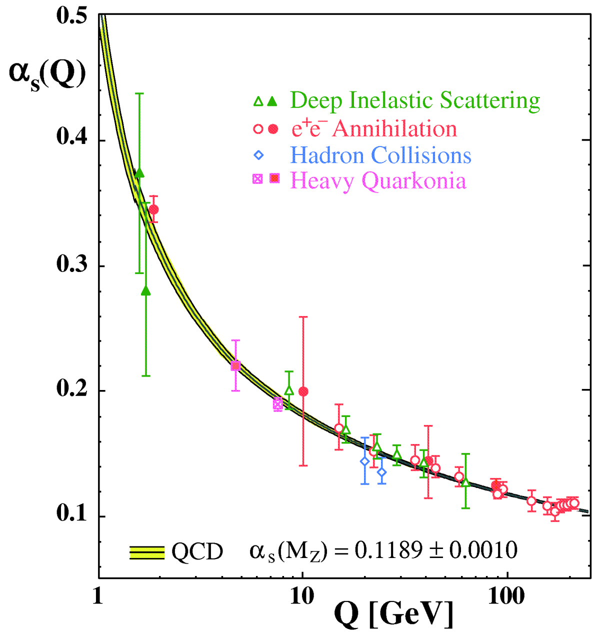

Figure 5: Running coupling constant of QCD from S. Bethke, Prog. Part. Nucl. Phys. **58**, 351 (2007)

Confinement

- Soft chromodynamic interactions, which occur at the momentum transfer that is not much greater than $\Lambda_{\rm QCD}$, are strong and cannot be described in terms of perturbation theory. Processes driven by such interactions are called non-perturbative.
- There are no universally applicable methods to describe non-perturbative phenomena.
- Confinement does not allow quarks and gluons to exist as free separate objects.
- More generally, the confinement does not allow for an existence of objects of non-vanishing color charge.
- Confinement is a non-perturbative phenomenon which has not been derived yet from QCD. It belongs to the Millennium Prize Problems. A correct solution will be awarded one million US dollars by the Clay Mathematics Institute.