Unified Electro-Weak Theory

In 1960s Sidney Glashow, Abdus Salam and Steven Weinberg formulated a unified theory of electromagnetic and weak interactions. The theory is now a part of the Standard Model which, as the whole model, is fully confirmed experimentally.

Matter particles

- The left handed leptons and neutrinos are grouped in the doublets

$$L_e \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \qquad L_\mu \equiv \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \qquad L_\tau \equiv \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \tag{1}$$

while the right handed leptons are singlets

$$R_e \equiv e_R, \qquad R_\mu \equiv \mu_R, \qquad R_\tau \equiv \tau_R. \tag{2}$$

- As discussed in detail in Lecture V, the left and right handed components of Dirac spinors are obtained as

$$\psi_L = P_L \psi, \qquad \psi_R = P_R \psi, \tag{3}$$

where the projection operators P_L and P_R are defined in the following way

$$P_L \equiv \frac{1}{2}(\mathbb{1} + \gamma^5), \qquad P_R \equiv \frac{1}{2}(\mathbb{1} - \gamma^5). \tag{4}$$

- The doublets should be understood as pairs of left-handed spinors that is

$$L_e \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \equiv \begin{pmatrix} P_L \psi_{\nu_e} \\ P_L \psi_e \end{pmatrix}. \tag{5}$$

- We will also need the Dirac conjugate spinor doublets like

$$\overline{L}_e \equiv (\overline{\nu}_e, \overline{e})_L \equiv (\overline{\psi}_{\nu_e} P_R, \overline{\psi}_e P_R). \tag{6}$$

- Analogously to the lepton sector, one groups the left handed quarks in the doublets

$$L_u \equiv \begin{pmatrix} u \\ d' \end{pmatrix}, \qquad L_c \equiv \begin{pmatrix} c \\ s' \end{pmatrix}, \qquad L_t \equiv \begin{pmatrix} t \\ b' \end{pmatrix}, \tag{7}$$

and the right hand u, c, t quarks in singlets

$$R_u \equiv u_R, \qquad R_c \equiv c_R, \qquad R_t \equiv t_R. \tag{8}$$

The 'rotated' quarks d', s', b' are obtained from the quarks d, s, b by means of the Kobayashi-Maskawa matrix which in case of two lightest flavors is the Cabbibo matrix

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta_c\, d + \sin\theta_c\, s \\ \cos\theta_c\, s - \sin\theta_c\, d \end{pmatrix}, \tag{9}$$

where $\theta_c \approx 13^\circ$ is the Cabibbo angle.

Weak isospin and weak hypercharge

- In analogy to the isospin known from strong interactions, one introduces the weak isospin T. The left-handed doublets have the isospin $T = 1/2$ while the right-handed singlets have $T = 0$. The isospin third component of the upper members of the left-handed doublets (neutrinos and quarks u, c, t) is $T_3 = 1/2$ while $T_3 = -1/2$ for the lower members of the left-handed doublets (leptons and quarks d', s', t').

Table 1: Electric charges, third components of the weak isospin and hypercharges

	Q	T_3	Y
neutrinos ν_e, ν_μ, ν_τ	0	$\frac{1}{2}$	-1
left-handed leptons e_L, μ_L, τ_L	-1	$-\frac{1}{2}$	-1
right-handed leptons e_R, μ_R, τ_R	-1	0	-2
left-handed quarks u_L, c_L, t_L	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
left-handed quarks d'_L, s'_L, b'_L	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$
right-handed quarks u_R, c_R, t_R	$\frac{2}{3}$	0	$\frac{4}{3}$
right-handed quarks d'_R, s'_R, b'_R	$-\frac{1}{3}$	0	$-\frac{2}{3}$

- The weak hypercharge Y is a quantum number relating the electric charge Q and the third component of weak isospin T_3 as
$$Y = 2(Q - T_3). \tag{10}$$
- The values of electric charges, third components of the weak isospin and hypercharges of leptons and quarks are collected in the Table 1.

Gauge fields

- Our objective is to construct a theory which unifies the electromagnetic and weak interactions. Once QED is the U(1) gauge theory, a presence of the U(1) symmetry in the unified theory is expected.
- Since the weak currents, which are known from the Fermi theory, are constructed out of the doublets (1) and (7), it is natural to assume that the SU(2) symmetry is involved here.
- The gauge group of the unified theory is the inner product of SU(2) and U(1) groups. The SU(2) transformations act on the doublets of weak isospin while the U(1) transformations act on all spinors.
- The group is often denoted as $\mathrm{SU(2)}_L$ to stress that the gauge transformations act on left-handed doublets.
- The SU(2) and U(1) gauge vector fields are denoted as W_a^μ with $a = 1, 2, 3$ and B^μ, respectively.
- The Lagrangian density of gauge fields is
$$\mathcal{L}_{\text{gauge}} = \frac{1}{4} W_a^{\mu\nu} W_{a\,\nu\mu} + \frac{1}{4} B^{\mu\nu} B_{\nu\mu}, \tag{11}$$
where
$$W_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu + g\epsilon^{abc} W_b^\mu W_c^\nu, \qquad B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu. \tag{12}$$
- The Lagrangian density (11) is invariant under the SU(2) and U(1) transformations of W^μ and B^μ, respectively. The transformations read
$$W^\mu \to U W^\mu U^\dagger - \frac{i}{g} (\partial^\mu U) U^\dagger, \qquad B^\mu \to B^\mu + \partial^\mu \Lambda, \tag{13}$$
where W and U are the matrices belonging to the fundamental representation of the SU(2) group.

Figure 1: Coupling of charged bosons $W^{\pm}$

Coupling of gauge fields to fermions

- We define two types of covariant derivatives:

$$D_L^\mu \equiv \partial^\mu - igW^\mu - \frac{i}{2}g'YB^\mu, \qquad D_R^\mu \equiv \partial^\mu - \frac{i}{2}g'YB^\mu. \tag{14}$$

The field W^μ is written in the fundamental representation that is $W^\mu = \tau^a W_a^\mu$ where τ^a with $a = 1, 2, 3$ are the generators of fundamental representation of SU(2) group.

- Since the gauge group is the inner product of two groups, there are two gauge couplings g and g' which are independent from each other.

- As we remember, $\tau^a = \frac{1}{2}\sigma^a$ where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

- The Lagrangian density which describes the interaction of fermions with gauge fields is

$$\mathcal{L}_{\text{gauge-fermion}} = \sum_f \Big(\overline{L}_f i\gamma_\mu D_L^\mu L_f + \overline{R}_f i\gamma_\mu D_R^\mu R_f \Big), \tag{16}$$

where the sum is taken over all lepton and quark flavors.

- Let us write down the first term of the Lagrangian (16), using the explicit form of the Pauli matrices (15). One finds

$$\overline{L}_f i\gamma_\mu D_L^\mu L_f = \overline{L}_f i\gamma_\mu \left(\partial^\mu - \frac{i}{2}g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{i}{2}g \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_2^\mu - \frac{i}{2}g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3^\mu - \frac{i}{2}g'Y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B^\mu \right) L_f$$

$$= \overline{L}_f i\gamma_\mu \left(\partial^\mu - \frac{i}{2}g \begin{pmatrix} 0 & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & 0 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} gW_3^\mu + g'YB^\mu & 0 \\ 0 & -gW_3^\mu + g'YB^\mu \end{pmatrix} \right) L_f. \tag{17}$$

- The structure of the first term in Eq. (17) suggests to introduce the charged gauge bosons

$$W_\pm^\mu = \frac{1}{\sqrt{2}} \left(W_1^\mu \pm iW_2^\mu \right), \tag{18}$$

which are coupled to the 'charged' currents like $\overline{\psi}_e \gamma^\mu (1 + \gamma^5) \psi_\nu$. The corresponding vertices of Feynman diagrams are shown in Fig. 1.

- Now, we combine the last diagonal term in Eq. (17) with the interaction part of $\overline{R}_f i\gamma_\mu D_R^\mu R_f$ and we get

$$\overline{L}_f i\gamma_\mu \left(-\frac{i}{2} \begin{pmatrix} gW_3^\mu + g'YB^\mu & 0 \\ 0 & -gW_3^\mu + g'YB^\mu \end{pmatrix} \right) L_f + \overline{R}_f i\gamma_\mu \Big(-\frac{i}{2} g'YB^\mu \Big) R_f$$

$$= gW_3^\mu J_\mu^3 + \frac{1}{2} g' B^\mu J_\mu^Y, \tag{19}$$

where J_μ^3 and J_μ^Y are the weak neutral and hypercharge currents which in case of a single lepton l and its neutrino ν are

$$J_\mu^3 = \frac{1}{2} (\overline{\nu}_L \gamma_\mu \nu_L - \bar{l}_L \gamma_\mu l_L), \tag{20}$$

$$J_\mu^Y = -\overline{\nu}_L \gamma_\mu \nu_L - \bar{l}_L \gamma_\mu l_L - 2\bar{l}_R \gamma_\mu l_R. \tag{21}$$

The notation is $l_L \equiv P_L \psi_l$, $l_R \equiv P_R \psi_l$, $\bar{l}_L \equiv \overline{\psi}_l P_R$, $\bar{l}_R \equiv \overline{\psi}_l P_L$ etc.

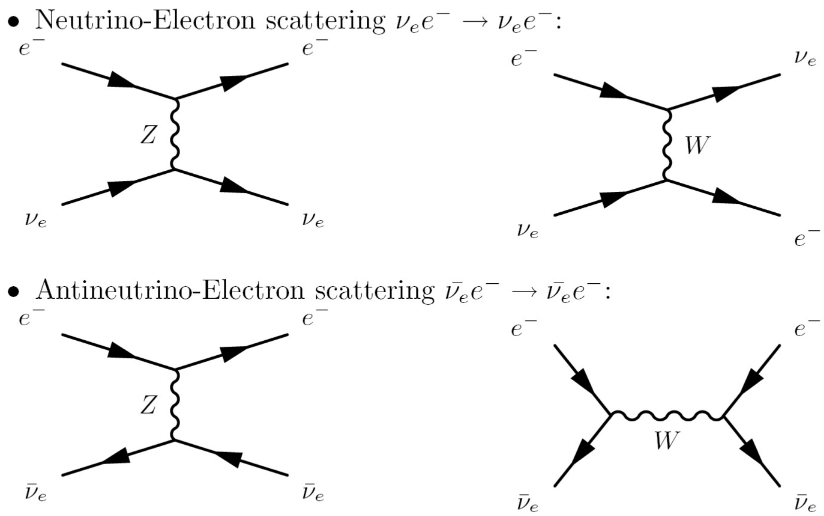

Figure 2: Elastic scattering of electron neutrinos and antineutrinos on electron

- On one observes that the electromagnetic current equals

$$J_\mu^{\rm em} \equiv -\bar{l}\gamma_\mu l = \frac{1}{2}J_\mu^Y + J_\mu^3, \tag{22}$$

where we have taken into account that the lepton electric charge is -1 and

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi}_L\gamma^\mu\psi_L + \overline{\psi}_R\gamma^\mu\psi_R. \tag{23}$$

- Using the relation (22), the right-hand-side of the equality (19) becomes

$$gW_3^\mu J_\mu^3 + \frac{1}{2}g'B^\mu J_\mu^Y = gW_3^\mu J_\mu^3 + g'B^\mu(J_\mu^{\rm em} - J_\mu^3) = (gW_3^\mu - g'B^\mu)J_\mu^3 + g'B^\mu J_\mu^{\rm em}. \tag{24}$$

- The last term in Eq. (24) may suggest to identify $g'B^\mu$ with eA^μ, where e is the elementary electric charge and A^μ is the electromagnetic potential. However, it cannot be right as Eq. (24) also shows that $g'B^\mu$ couples to the weak current $\overline{\nu}_L\gamma_\mu\nu_L$ which is electrically neutral. Therefore, we consider the 'rotated' fields

$$\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} B^\mu \\ W_3^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W\, B^\mu + \sin\theta_W\, W_3^\mu \\ \cos\theta_W\, W_3^\mu - \sin\theta_W\, B^\mu \end{pmatrix}, \tag{25}$$

where θ_W is called the Weinberg angle.

- Substituting $B^\mu = \cos\theta_W\, A^\mu - \sin\theta_W\, Z^\mu$ and $W_3^\mu = \cos\theta_W\, Z^\mu + \sin\theta_W\, A^\mu$ into the expression (24), one finds

$$\begin{aligned}(gW_3^\mu - g'B^\mu)J_\mu^3 + g'B^\mu J_\mu^{\rm em} &= \big((g\cos\theta_W - g'\sin\theta_W)Z^\mu - (g'\cos\theta_W - g\sin\theta_W)A^\mu\big)J_\mu^3 \\ &+ g'(\cos\theta_W\, A^\mu - \sin\theta_W\, Z^\mu)J_\mu^{\rm em}.\end{aligned} \tag{26}$$

One sees that we have to demand $g'\cos\theta_W - g\sin\theta_W = 0$ to cancel out the coupling of A^μ to J_μ^3 which can be electrically neutral. Therefore, the Weinberg angle is given as

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}. \tag{27}$$

- Now, we write the expression (26) as

$$(gW_3^\mu - g'B^\mu)J_\mu^3 + g'B^\mu J_\mu^{\rm em} = \big((g\cos\theta_W - g'\sin\theta_W)J_\mu^3 - g'\sin\theta_W\, J_\mu^{\rm em}\big)Z^\mu + g'\cos\theta_W\, A^\mu J_\mu^{\rm em}. \tag{28}$$

- The last term of the expression (28) shows that the electric elementary charge is

$$e = g' \cos\theta_W = g \sin\theta_W = \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{29}$$

- As we remember, the charged bosons $W^{\pm}_{\mu}$ are coupled to the charged weak currents which include the vector and axial-vector contributions. The coupling to the vector and axial-vector currents is the same. Eq. (28) shows that this is not the case for the neutral boson Z. Since J^3_μ contains the vector and axial-vector contributions but $J^{\rm em}_\mu$ only the vector one, the couplings to the vector and axial vector currents are

$$g_V = (g\cos\theta_W - g'\sin\theta_W)T_3 - 2g'\sin\theta_W, \qquad g_A = (g\cos\theta_W - g'\sin\theta_W)T_3. \tag{30}$$

- The weak neutral currents were discovered in 1973 through an analysis of the elastic scattering of electron neutrinos and antineutrinos on electron. The Feynman diagrams of the two processes are shown in Fig. 2.
- The relations (30) were experimentally confirmed.
- The measured value of the Weinberg angle is approximately $\theta_W = 29°$.

Gauge invariance and masses of fermions

- The Lagrangian density (16) is invariant under the SU(2) transformations of the fields W^μ and L_f and under the U(1) transformations of B^μ, L_f and R_f. The transformations read

$$W^\mu \to UW^\mu U^\dagger - \frac{i}{g}(\partial^\mu U)U^\dagger, \qquad L_f \to UL_f, \tag{31}$$

$$B^\mu \to B^\mu + \partial^\mu\Lambda, \qquad L_f \to e^{\frac{i}{2}g'Y\Lambda}L_f, \qquad R_f \to e^{\frac{i}{2}g'Y\Lambda}R_f, \tag{32}$$

where U are, as previously, the matrices belonging to the fundamental representation of the SU(2) group.

- We note that the fermions must be massless as the mass terms would violate the gauge invariance. The point is that the mass term of the fermion Lagrangian can be written as

$$\mathcal{L}_{\rm mass} = -m\,\overline{\psi}\psi = -m\,(\overline{\psi}_L + \overline{\psi}_R)(\psi_L + \psi_R) = -m\,(\overline{\psi}_L\psi_L + \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L + \overline{\psi}_R\psi_R). \tag{33}$$

Since $\overline{\psi}_L\psi_L = \overline{\psi}P_R P_L\psi = 0$ and $\overline{\psi}_R\psi_R = \overline{\psi}P_L P_R\psi = 0$, we have

$$\mathcal{L}_{\rm mass} = -m\,\overline{\psi}\psi = -m\,(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L). \tag{34}$$

Therefore, the mass term mixes up the left-handed and right-handed fermions, and consequently, it violates the SU(2) gauge symmetry.

- As we will see later on, the masses of leptons and quarks can be generated by means of the Higgs mechanism.
- The gauge invariant theory defined by the sum of Lagrangians (11) and (16) describes massless fermions interacting with four massless vector bosons. The theory is rather far from the known matter fermions experiencing the weak interactions.

Higgs field

- The Lagrangian density of the Higgs field is chosen as

$$\mathcal{L}_{\rm Higgs} = \big(D^\mu_L\Phi\big)^\dagger D_{L\,\mu}\Phi - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2, \tag{35}$$

where, as previously, $D^\mu_L \equiv \partial^\mu - igW^\mu - \frac{i}{2}g'YB^\mu$ and Φ is the two-component scalar field

$$\Phi = \begin{pmatrix}\phi_+\\ \phi_0\end{pmatrix}. \tag{36}$$

The fields ϕ_+ and ϕ_0 are complex and the subscripts refer to the electric charge. The hypercharge of both ϕ_+ and ϕ_0 is $Y = 1$.

- The Lagrangian density (35) is invariant under the SU(2) transformations of the fields W^μ and Φ and under the U(1) transformations of B^μ, ϕ_+ and Φ_0. The transformations read

$$W^\mu \to U W^\mu U^\dagger - \frac{i}{g}(\partial^\mu U)U^\dagger, \qquad \Phi \to U\Phi, \tag{37}$$

$$B^\mu \to B^\mu + \partial^\mu \Lambda, \qquad \phi_+ \to e^{i\frac{i}{2}g'\Lambda}\phi_+, \qquad \phi_0 \to e^{i\frac{i}{2}g'\Lambda}\phi_0. \tag{38}$$

- Since the mass parameter μ^2 and the coupling constant λ obey $\mu^2 < 0$ and $\lambda > 0$, the SU(2) symmetry is spontaneously broken.

- We choose the field which minimizes the potential energy as

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v \equiv \sqrt{-\frac{\mu^2}{\lambda}}. \tag{39}$$

  The coefficient $1/\sqrt{2}$ is introduced to get the standard factor $\frac{1}{2}$ in front of the kinetic term of the Higgs field in the Lagrangian (53).

- Since the theory we construct has to preserve the electromagnetic U(1) symmetry, the ground state or vacuum (39) must be symmetric. It is indeed the case as the field ϕ_0 is chosen to be electrically neutral.

- As discussed in Lecture VI, we can define the real scalar field $H(x)$ as

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H - v \end{pmatrix}, \tag{40}$$

  which substituted into the Lagrangian (35) immediately gives

$$\mu^2 \Phi^\dagger \Phi = \frac{\mu^2}{2}(H-v)^2, \tag{41}$$

$$\lambda(\Phi^\dagger\Phi)^2 = \frac{\lambda}{4}(H-v)^4. \tag{42}$$

- The term $\left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi$ is computed in the following way

$$\begin{aligned}
\left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi &= \frac{1}{2}(0, H-v)\big(\overleftarrow{\partial_\mu} + igW_\mu + \frac{i}{2}g'B_\mu\big)\big(\partial^\mu - igW^\mu - \frac{i}{2}g'B^\mu\big)\begin{pmatrix} 0 \\ H-v \end{pmatrix} \\
&= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{ig}{2}(0, \partial_\mu H)W^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} - \frac{ig'}{4}(0, \partial_\mu H)B^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} \\
&\quad + \frac{ig}{2}(0, H-v)W_\mu\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} + \frac{ig'}{4}(0, H-v)B_\mu\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} \\
&\quad + \frac{g^2}{2}(0, H-v)W_\mu W^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} + \frac{gg'}{4}(0, H-v)W_\mu B^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} \\
&\quad + \frac{g'g}{4}(0, H-v)B_\mu W^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} + \frac{g'^2}{8}(0, H-v)B_\mu B^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix}.
\end{aligned} \tag{43}$$

- One finds that the second and fourth terms in Eq. (43) cancel each other because

$$(0, \partial_\mu H)W^\mu\begin{pmatrix} 0 \\ H-v \end{pmatrix} = \frac{1}{2}(\partial_\mu H)(H-v)W_a^\mu\,(0,1)\sigma_a\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0, H-v)W_\mu\begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix}. \tag{44}$$

- The third and the fifth terms in Eq. (43) also cancel each other.

- The sixth term in Eq. (43) is computed as

$$\frac{g^2}{2}(0, H-v)W_\mu W^\mu \begin{pmatrix} 0 \\ H-v \end{pmatrix} = \frac{g^2}{8}(H-v)^2 W_{a\mu} W_b^\mu \quad (0,1)\sigma^a\sigma^b \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{45}$$

Using the explicit for of Pauli matrices (15), one finds that

$$\frac{g^2}{2}(0, H-v)W_\mu W^\mu \begin{pmatrix} 0 \\ H-v \end{pmatrix} = \frac{g^2}{8}(H-v)^2 W_{a\mu} W_a^\mu. \tag{46}$$

- The last term in Eq. (43) is

$$\frac{g^2}{8}(0, H-v)B_\mu B^\mu \begin{pmatrix} 0 \\ H-v \end{pmatrix} = \frac{g^2}{8}(H-v)^2 B_\mu B^\mu (0,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{g^2}{8}(H-v)^2 B_\mu B^\mu. \tag{47}$$

- The sum of the seventh and eighth terms from Eq. (43) equals

$$\frac{gg'}{4}(0, H-v)(W_\mu B^\mu + B_\mu W^\mu)\begin{pmatrix} 0 \\ H-v \end{pmatrix} = \frac{gg'}{4}(H-v)^2 W_a^\mu B_\mu (0,1)\sigma^a \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{gg'}{4}(H-v)^2 W_3^\mu B_\mu. \tag{48}$$

- So, the final result of the term $\left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi$ is

$$\left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2}{8}(H-v)^2 W_{a\mu} W_a^\mu + \frac{g'^2}{8}(H-v)^2 B_\mu B^\mu - \frac{gg'}{4}(H-v)^2 W_3^\mu B_\mu. \tag{49}$$

- Let us now introduce the physical fields $W_\pm^\mu, Z^\mu, A^\mu$. Since

$$\begin{aligned} W_{a\mu} W_a^\mu &= 2W_+^\mu W_{-\,\mu} + W_3^\mu W_{3\,\mu}, \\ B^\mu &= \cos\theta_W\, A^\mu - \sin\theta_W\, Z^\mu, \\ W_3^\mu &= \cos\theta_W\, Z^\mu + \sin\theta_W\, A^\mu, \end{aligned}$$

the expression (49) is rewritten as

$$\begin{aligned} \left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi &= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2}{4}(H-v)^2 W_+^\mu W_{-\,\mu} \\ &+ \frac{g^2}{8}(H-v)^2 W_3^\mu W_{3\,\mu} + \frac{g'^2}{8}(H-v)^2 B_\mu B^\mu - \frac{gg'}{4}(H-v)^2 W_3^\mu B_\mu \\ &= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2}{4}(H-v)^2 W_+^\mu W_{-\,\mu} \\ &+ \frac{1}{8}(H-v)^2\Big[\big(g^2\cos^2\theta_W + g'^2\sin^2\theta_W + 2gg'\cos\theta_W\sin\theta_W\big)Z^\mu Z_\mu \\ &\qquad + \big(g^2\sin^2\theta_W + g'^2\cos^2\theta_W - 2gg'\cos\theta_W\sin\theta_W\big)A^\mu A_\mu \\ &\qquad + 2\big(g^2\sin\theta_W\cos\theta_W - g'^2\sin\theta_W\cos\theta_W - gg'(\cos^2\theta_W - \sin^2\theta_W)\big)Z^\mu A_\mu\Big]. \end{aligned} \tag{50}$$

- Using the formulas (27), one finds that the coefficients in front of $A^\mu A_\mu$ and $Z^\mu A_\mu$ in Eq. (50) vanish, and finally one obtains

$$\left(D_L^\mu \Phi\right)^\dagger D_{L\,\mu}\Phi = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2}{4}(H-v)^2 W_+^\mu W_{-\,\mu} + \frac{1}{8}(g^2+g'^2)(H-v)^2 Z^\mu Z_\mu. \tag{51}$$

- Combing the results (41), (42) and (50), the Higgs Lagrangian (35) becomes

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2}{4}(H-v)^2 W_+^\mu W_{-\,\mu} + \frac{1}{8}(g^2+g'^2)(H-v)^2 Z^\mu Z_\mu - \frac{\mu^2}{2}(H-v)^2 - \frac{\lambda}{4}(H-v)^4, \tag{52}$$

which is finally rewritten as

$$\mathcal{L}_{\rm Higgs} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2}m_H^2 H^2 - \frac{\lambda}{4}(H^4 - 4vH^3) \tag{53}$$

$$- m_W^2 W_+^\mu W_{-\,\mu} + \frac{g^2}{4}(H^2 - 2vH)W_+^\mu W_{-\,\mu} \tag{54}$$

$$- \frac{1}{2}m_Z^2 Z^\mu Z_\mu + \frac{1}{8}(g^2 + g'^2)(H^2 - 2vH)Z^\mu Z_\mu, \tag{55}$$

where the masses are

$$m_H^2 = -2\mu^2, \qquad m_W^2 = \frac{1}{4}g^2 v^2 = -\frac{g^2\mu^2}{4\lambda}, \qquad m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2 = -\frac{(g^2+g'^2)\mu^2}{4\lambda}. \tag{56}$$

- The experimentally obtained masses are

$$m_H \approx 125 \;\; {\rm GeV}/c^2, \qquad m_W \approx 80 \;\; {\rm GeV}/c^2, \qquad m_Z \approx 91 \;\; {\rm GeV}/c^2. \tag{57}$$

- The first line of Eq. (53) gives the kinetic and mass terms of the Higgs field and its self interaction. The second line provides the mass term of charged bosons $W^\pm$ and their interaction with the Higgs bosons. In the third line we see the mass term of neutral bosons Z^0 and their interaction with the Higgs bosons.

- One observes that

$$\frac{m_W}{m_Z} = \frac{g^2}{g^2 + g'^2} = \cos\theta_W, \tag{58}$$

which is the well-know prediction of the Standard Model.

Masses of fermions and Yukawa couplings

- Matter particles of the theory we construct step by step are still massless. As already mentioned, the standard mass term of the fermion Lagrangian which is

$$\mathcal{L}_{\rm mass} = -m\,\overline{\psi}\psi = -m\,(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L), \tag{59}$$

mixes up the left-handed and right-handed fermions, and consequently, it violates the $\mathrm{SU}(2)_L$ gauge symmetry.

- The masses of leptons and quarks can be introduced by means of the Higgs mechanism not violating the $\mathrm{SU}(2)_L$ gauge symmetry.

- One postulates the Yukawa coupling of leptons to the Higgs field Φ as

$$\mathcal{L}_{\rm Yukawa} = -\sum_l G_l\big[\overline{R}_l(\Phi^\dagger L_l) + (\overline{L}_l\Phi)R_l\big], \tag{60}$$

where G_l is the dimensionless coupling constant. The Lagrangian (61) is invariant under the $\mathrm{SU}(2)_L$ and U(1) transformation.

- Due to the spontaneous symmetry breakdown, the Higgs field can be parameterized by the formula (40) which substituted in Eq. (61) gives

$$\mathcal{L}_{\rm Yukawa} = \sum_l \frac{G_l}{\sqrt{2}}(H - v)\big[\bar{l}_R l_L + \bar{l}_L R_l\big] = \sum_l \frac{G_l}{\sqrt{2}}(H - v)\overline{\psi}_l\psi_l \tag{61}$$

$$= \sum_l \Big(- m_l\overline{\psi}_l\psi_l + \frac{G_l}{\sqrt{2}}H\overline{\psi}_l\psi_l\Big),$$

where the lepton mass is $m_l = \frac{G_l v}{\sqrt{2}}$. We see that neutrinos remain massless because of the vanishing upper component of the Higgs field (40).

- A generation of quark masses is a more complex issue even so a general idea is similar to that of the lepton case. First of all, quarks of all flavors (not only d, s, b) should acquire a mass. Therefore, the Higgs doublet must be rearranged. A more severe complication is that the 'rotated' quarks, which enter the weak currents, are not the mass eigenstates. Therefore, the Cabibbo-Kobayashi-Maskawa matrix, which mixes up quarks d, s, b is involved in the Yukawa couplings of quarks to Higgs.

Electro-weak Lagrangian

- The complete electro-weak Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{gauge+Higgs}} + \mathcal{L}_{\text{leptons+Yukawa}} + \mathcal{L}_{\text{quarks+Yukawa}}. \tag{62}$$

- The Lagrangian of gauge and Higgs fields is

$$\begin{aligned} \mathcal{L}_{\text{gauge+Higgs}} &= \frac{1}{4}F^{\mu\nu}F_{\nu\mu} \\ &+ \frac{1}{2}W_+^{\mu\nu}W_{-\nu\mu} + m_W^2 W_+^{\mu}W_{-\mu} \\ &+ \frac{1}{4}Z^{\mu\nu}Z_{\nu\mu} + \frac{1}{2}m_Z^2 Z^{\mu}Z_{\mu} \\ &+ \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2}m_H^2 H^2 \\ &+ W^+W^-W^+W^- \quad + \quad W^+W^-\gamma \quad + \quad W^+W^-\gamma\gamma \\ &+ W^+W^-\gamma Z^0 \quad + \quad W^+W^-Z^0 \quad + \quad W^+W^-Z^0Z^0 \\ &+ W^+W^-H \quad + \quad W^+W^-HH \\ &+ Z^0Z^0H \quad + \quad Z^0Z^0HH \\ &+ HHH \quad + \quad HHHH, \end{aligned} \tag{63}$$

where the interaction terms are represented only symbolically and

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad W_\pm^{\mu\nu} \equiv \partial^\mu W_\pm^\nu - \partial^\nu W_\pm^\mu, \qquad Z^{\mu\nu} \equiv \partial^\mu Z^\nu - \partial^\nu Z^\mu. \tag{64}$$

- The couplings among the gauge bosons A^μ, $W_\pm^\mu$ and Z^μ appear when the fields W_a^μ and B^μ expressed through the physical fields are substituted into the Lagrangian (11).

- The Lagrangian of lepton fields and their Yukawa couplings is

$$\begin{aligned} \mathcal{L}_{\text{leptons+Yukawa}} &= \sum_{l=e,\mu,\tau} \bar{l}(i\gamma^\mu\partial_\mu - m_l)l \;+ \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau} \overline{\nu} i\gamma^\mu\partial_\mu\nu \\ &+ \;\bar{l}\,l\,\gamma \;+\; \overline{\nu}_l\,l\,W^+ \;+\; \bar{l}\,\nu_l\,W^- \;+\; \bar{l}\,l\,Z \;+\; \overline{\nu}_l\,\nu_l\,Z + \;+\; \bar{l}\,l\,H\,H. \end{aligned} \tag{65}$$

- The Lagrangian of quark fields and their Yukawa couplings is

$$\begin{aligned} \mathcal{L}_{\text{leptons+Yukawa}} &= \sum_{q=u,d,c,s,t,b} \overline{q}(i\gamma^\mu\partial_\mu - m_q)q \\ &+ \;\overline{u}\,d'\,W^+ \;+\; \overline{c}\,s'\,W^+ \;+\; \overline{t}\,b'\,W^+ \\ &+ \;\overline{d}'\,u\,W^- \;+\; \overline{s}'\,c\,W^- \;+\; \overline{b}'\,t\,W^- \\ &+ \;\overline{q}\,q\,\gamma \;+\; \overline{q}\,q\,Z^0 \;+\; \overline{q}\,q\,H. \end{aligned} \tag{66}$$