Functional Methods in Vacuum QFT

In the previous lectures we have been using the operator formalism of quantum field theory. Now, we are going to introduce the functional methods in statistical QFT which allow one to derive a perturbative expansion of Green's functions in an elegant and economical way. However, we start with a reminder of the functional methods in vacuum QFT

Generating functional

A key object of the functional methods of QFT is the generating functional which is used to obtain various Green's function and their perturbative expansion. The functional offers a rather simple way to prove the Wick theorem.

Preliminaries

- We consider, as previously, the real scalar field ϕ, starting with the field interacting only with an external source J. The self-interacting field will be discussed further on. The Lagrangian density, which is quadratic in the field, is

$$\mathcal{L} = \mathcal{L}_0 + J(x)\phi(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) + J(x)\phi(x), \tag{1}$$

where the Lagrangian density of free field reads

$$\mathcal{L}_0(x) \equiv \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x), \tag{2}$$

and m is the mass. The equation of motion is

$$[\partial_\mu\partial^\mu + m^2]\,\phi(x) = J(x). \tag{3}$$

- The action of free field equals

$$S_0[\phi] = \frac{1}{2}\int d^4x\mathcal{L}_0 = \int d^4x(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \tag{4}$$

but it can be rewritten as

$$S_0[\phi] = \frac{1}{2}\int d^4x\, d^4y\,\phi(x)[-\partial_\mu\partial^\mu - m^2]\delta(x-y)\phi(y). \tag{5}$$

We have performed the partial integration and neglected the surface term because $\phi(x_0 \to \pm\infty, |\mathbf{x}| \to \infty) = 0$.

- The Hamiltonian density is defined through the Legendre transformation

$$\mathcal{H}(x) = \pi(x)\,\dot{\phi}(x) - \mathcal{L}(x), \tag{6}$$

and the conjugate momentum is

$$\pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\phi}(x)} = \dot{\phi}(x). \tag{7}$$

- The Hamiltonian and the Hamiltonian density are

$$H(t) = \int d^3x\,\mathcal{H}(x), \tag{8}$$

$$\mathcal{H}(x) = \underbrace{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi^2(x)}_{\equiv\mathcal{H}_0(x)} - \phi(x)\,J(x). \tag{9}$$

- Our aim here is to derive the path integral representation of the generating functional $W_0[J]$ which is defined as the vacuum-vacuum transition amplitude in the presence of interaction $\phi(x)J(x)$. Therefore,
$$W_0[J] \equiv \langle 0 \text{ out}|0 \text{ in}\rangle, \tag{10}$$
where $|0 \text{ in}\rangle$ is the vacuum state in the remote past and $|0 \text{ out}\rangle$ in the remote future. Both states are in the Heisenberg picture.

- The generating functional, which is computed with the Hamiltonian (8), is called 'free' and is labeled with the subscript '0' even so the field interacts with the external source J.

- Now, we express the vacuum states in the Heisenberg picture through the vacuum state in the Schrödinger picture at an arbitrarily chosen time $t = 0$ which is denoted as $|0\rangle$. Thus, we get
$$|0 \text{ out}\rangle = T \exp\Big[i \int_0^\infty \hat{H}(t)dt\Big]|0\rangle, \tag{11}$$
$$|0 \text{ in}\rangle = T \exp\Big[i \int_0^{-\infty} \hat{H}(t)dt\Big]|0\rangle = T \exp\Big[-i \int_{-\infty}^0 \hat{H}(t)dt\Big]|0\rangle, \tag{12}$$
where $\hat{H}$ is the Hamiltonian (8) which includes the interaction term $\phi(x)J(x)$. We clearly distinguish operators, which act in the Hilbert space of states, from their classical counterparts as the operators are denoted with hats.

- Keeping in mind that
$$\langle 0 \text{ out}| = (|0 \text{ out}\rangle)^\dagger = \langle 0|T \exp\Big[-i \int_0^\infty \hat{H}(t)dt\Big], \tag{13}$$
the transition amplitude in the Schrödinger picture equals
$$\boxed{W_0[J] = \langle 0|T \exp\Big[-i \int_{-\infty}^\infty \hat{H}(t)dt\Big]|0\rangle,} \tag{14}$$
which can be rewritten as
$$W_0[J] = \langle 0|T \exp\Big[-i \int d^4x \big(\mathcal{H}_0(x) - \phi(x)\, J(x)\big)\Big]|0\rangle. \tag{15}$$

- As it is well known, the $n-$point Green's function equals the $n-$th order functional derivative of the generating functional (15)
$$i\Delta_0^{(n)}(x_1, \ldots, x_n) = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W_0[J]\Big|_{J=0}, \tag{16}$$
which is actually the reason why $W_0[J]$ is called the 'generating functional'.

- For our purpose it is more appropriate the write down the generating functional (14) as
$$W_0[J] = \lim_{t\to\infty} \langle 0|\hat{U}(t, -t)|0\rangle, \tag{17}$$
where $\hat{U}(t, -t)$ is the evolution operator
$$\hat{U}(t, -t) = T \exp\Big[-i \int_{-t}^t \hat{H}(t')dt'\Big]. \tag{18}$$

Path-integral representation of free generating functional

- We are going to derive the path-integral representation of the functional (17).
- Performing the discretization of time, the generating functional (17) is written as

$$W_0[J] = \lim_{t\to\infty}\lim_{N\to\infty}\langle 0|T\exp\Big[-i\sum_{k=0}^{N-1}\hat{H}(t_k)\Delta t\Big]|0\rangle \tag{19}$$
$$= \lim_{t\to\infty}\lim_{N\to\infty}\langle 0|\exp\big[-i\hat{H}(t_{N-1})\Delta t\big]\exp\big[-i\hat{H}(t_{N-2})\Delta t\big]\cdots\exp\big[-i\hat{H}(t_0)\Delta t\big]|0\rangle,$$

  where $t_k = -t + k\Delta t$ and $\Delta t \equiv 2t/N$.

- Using the complete set of states $|\phi_k(\mathbf{x})\rangle$, where $\phi_k(\mathbf{x}) \equiv \phi(t_k,\mathbf{x})$, the generating functional equals

$$W_0[J] = \lim_{t\to\infty}\lim_{N\to\infty}\int\mathcal{D}\phi_{N-1}(\mathbf{x})\int\mathcal{D}\phi_{N-2}(\mathbf{x})\cdots\int\mathcal{D}\phi_2(\mathbf{x})\int\mathcal{D}\phi_1(\mathbf{x}) \tag{20}$$
$$\times\ \langle 0|\exp\big[-i\hat{H}(t_{N-1})\Delta t\big]|\phi_{N-1}(\mathbf{x})\rangle\langle\phi_{N-1}(\mathbf{x})|\exp\big[-i\hat{H}(t_{N-2})\Delta t\big]|\phi_{N-2}(\mathbf{x})\rangle$$
$$\cdots\langle\phi_2(\mathbf{x})|\exp\big[-i\hat{H}(t_1)\Delta t\big]|\phi_1(\mathbf{x})\rangle\langle\phi_1(\mathbf{x})|\exp\big[-i\hat{H}(t_0)\Delta t\big]|0\rangle,$$

  where we have the functional integrals over the fields $\phi_k(\mathbf{x})$ at a given time t_k.

- Let us now compute

$$\langle\phi_{k+1}(\mathbf{x})|\exp\big[-i\hat{H}(t_k)\Delta t\big]|\phi_k(\mathbf{x})\rangle. \tag{21}$$

- Expanding the exponent

$$\exp\big[-i\hat{H}(t_k)\Delta t\big] = 1 - i\hat{H}(t_k)\Delta t + \mathcal{O}(\Delta t^2) \tag{22}$$

  and taking into account only the linear term, we have

$$\langle\phi_{k+1}(\mathbf{x})|\exp\big[-i\hat{H}(t_k)\Delta t\big]|\phi_k(\mathbf{x})\rangle = \delta\big[\phi_{k+1}(\mathbf{x})-\phi_k(\mathbf{x})\big] - i\Delta t\langle\phi_{k+1}(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle, \tag{23}$$

  where we have taken into account that

$$\langle\phi_{k+1}(\mathbf{x})|\phi_k(\mathbf{x})\rangle = \delta\big[\phi_{k+1}(\mathbf{x}) - \phi_k(\mathbf{x})\big] \tag{24}$$

  with $\delta\big[\phi_{k+1}(\mathbf{x}) - \phi_k(\mathbf{x})\big]$ being the functional (infinitely dimensional) Dirac delta. When the space variable $\mathbf{x}$ is discretized and the space points are numerated by the index $i = 1, 2, \dots N$, the state $|\phi_k(\mathbf{x})\rangle$ is characterized by the set of field values $\{\phi_k^1, \phi_k^2, \dots \phi_k^N\}$ where $\phi_k^i \equiv \phi_k(\mathbf{x}_i)$.

- We write

$$|\phi_k(\mathbf{x})\rangle = \lim_{N\to\infty}|\phi_k^1, \phi_k^2, \dots \phi_k^N\rangle \tag{25}$$

  and the functional Dirac delta should be understood as

$$\delta\big[\phi_{k+1}(\mathbf{x}) - \phi_k(\mathbf{x})\big] = \lim_{N\to\infty}\Pi_{i=1}^N\,\delta\big(\phi_{k+1}^i - \phi_k^i\big). \tag{26}$$

- The problem is now reduced to compute $\langle\phi_{k+1}(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle$.

- We introduce the set of complete momentum eigenstates $|\pi_k(\mathbf{x})\rangle$. Then,

$$\langle\phi_{k+1}(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle = \int \frac{\mathcal{D}\pi_k(\mathbf{x})}{2\pi}\ \langle\phi_{k+1}(\mathbf{x})|\pi_k(\mathbf{x})\rangle\langle\pi_k(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle. \tag{27}$$

- Performing the discretization of space variable $\mathbf{x}$ such that the index $i = 1,\, 2, \dots N$ numerates the space points, we first compute

$$\begin{aligned}\langle\phi_{k+1}(\mathbf{x})|\pi_k(\mathbf{x})\rangle &= \lim_{N\to\infty}\langle\phi^N_{k+1},\phi^{N-1}_{k+1},\dots\phi^1_{k+1}|\pi^1_k,\pi^2_k,\dots\pi^N_k\rangle \\ &= \lim_{N\to\infty}\exp\Big[i\Delta V\sum_{i=1}^N \pi^i_k\phi^i_{k+1}\Big] = \exp\Big[i\int d^3x\ \pi_k(\mathbf{x})\,\phi_{k+1}(\mathbf{x})\Big],\end{aligned} \tag{28}$$

where ΔV is the volume of elementary cube in the discretized R^3-space.

- Using the Hamiltonian (8) we compute $\langle\pi_k(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle$ as

$$\begin{aligned}&\langle\pi_k(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle \\ &= \langle\pi_k(\mathbf{x})|\int d^3x'\Big[\frac{1}{2}\hat{\pi}^2_k(\mathbf{x}') + \frac{1}{2}\big(\nabla\hat{\phi}_k(\mathbf{x}')\big)^2 + \frac{1}{2}m^2\hat{\phi}^2_k(\mathbf{x}') - \hat{\phi}_k(\mathbf{x}')\,J_k(\mathbf{x}')\Big]|\phi_k(\mathbf{x})\rangle.\end{aligned} \tag{29}$$

- It is not difficult to guess the result but it is instructive to analyze the problem in detail by performing the discretization. Then, the first term is

$$\begin{aligned}\frac{1}{2}\langle\pi_k(\mathbf{x})|\int d^3x\,\hat{\pi}^2_k(\mathbf{x})|\phi_k(\mathbf{x})\rangle &= \lim_{N\to\infty}\frac{\Delta V}{2}\langle\pi^N_k,\pi^{N-1}_k,\dots\pi^1_k|\sum_{i=1}^N(\hat{\pi}^i_k)^2|\phi^1_k,\phi^2_k,\dots\phi^N_k\rangle \\ &= \lim_{N\to\infty}\frac{\Delta V}{2}\Big[\sum_{i=1}^N(\pi^i_k)^2\Big]\langle\pi^N_k,\pi^{N-1}_k,\dots\pi^1_k|\phi^1_k,\phi^2_k,\dots\phi^N_k\rangle \\ &= \lim_{N\to\infty}\frac{\Delta V}{2}\Big[\sum_{i=1}^N(\pi^i_k)^2\Big]\exp\Big[-i\Delta V\sum_{i=1}^N\pi^i_k\phi^i_k\Big] \\ &= \frac{1}{2}\int d^3x\,\pi^2_k(\mathbf{x})\exp\Big[-i\int d^3x\ \pi_k(\mathbf{x})\,\phi_k(\mathbf{x})\Big],\end{aligned} \tag{30}$$

where we have used the fact that $|\pi^1_k,\pi^2_k,\dots\pi^N_k\rangle$ is the momentum eigenstate and thus

$$\langle\pi^N_k,\pi^{N-1}_k,\dots\pi^1_k|\hat{\pi}^i_k = \pi^i_k\langle\pi^N_k,\pi^{N-1}_k,\dots\pi^1_k|. \tag{31}$$

- Computing the remaining terms from Eq. (29) - the gradient term actually requires some extra care - one finds

$$\langle\pi_k(\mathbf{x})|\hat{H}(t_k)|\phi_k(\mathbf{x})\rangle = H(t_k)\ \exp\Big[-i\int d^3x\ \pi_k(\mathbf{x})\,\phi_k(\mathbf{x})\Big]. \tag{32}$$

- Combining the results (24) and (32), we get

$$
\begin{aligned}
\langle\phi_{k+1}(\mathbf{x})|\exp\big[-i\hat{H}(t_k)\Delta t\big]|\phi_k(\mathbf{x})\rangle &= \delta\big[\phi_{k+1}(\mathbf{x})-\phi_k(\mathbf{x})\big] \\
&\quad - i\Delta t\int\frac{\mathcal{D}\pi_k(\mathbf{x})}{2\pi}\;H(t_k)\;\exp\Big[-i\int d^3x\,\pi_k(\mathbf{x})\big(\phi_k(\mathbf{x})-\phi_{k+1}(\mathbf{x})\big)\Big]+\mathcal{O}(\Delta t^2) \\
&= \int\frac{\mathcal{D}\pi_k(\mathbf{x})}{2\pi}\;\exp\Big[-i\int d^3x\,\pi_k(\mathbf{x})\big(\phi_k(\mathbf{x})-\phi_{k+1}(\mathbf{x})\big)\Big]\Big(1-iH(t_k)\;\Delta t\Big)+\mathcal{O}(\Delta t^2) \\
&= \int\frac{\mathcal{D}\pi_k(\mathbf{x})}{2\pi}\;\exp\Big[i\Delta t\int d^3x\Big[\pi_k(\mathbf{x})\Big(\frac{\phi_{k+1}(\mathbf{x})-\phi_k(\mathbf{x})}{\Delta t}\Big)-\mathcal{H}(t_k,\mathbf{x})\Big]\Big]+\mathcal{O}(\Delta t^2),
\end{aligned}
\tag{33}
$$

  where we have used the functional identity

$$
\delta\big[\phi_{k+1}(\mathbf{x})-\phi_k(\mathbf{x})\big]=\int\frac{\mathcal{D}\pi_k(\mathbf{x})}{2\pi}\;\exp\Big[-i\int d^3x\,\pi_k(\mathbf{x})\big(\phi_k(\mathbf{x})-\phi_{k+1}(\mathbf{x})\big)\Big].
\tag{34}
$$

- Plugging the expression (33) into Eq. (20), we obtain

$$
\begin{aligned}
W_0[J] \;&=\; \lim_{t\to\infty}\lim_{N\to\infty}\int\mathcal{D}\phi_{N-1}(\mathbf{x})\int\mathcal{D}\phi_{N-2}(\mathbf{x})\cdots\int\mathcal{D}\phi_1(\mathbf{x}) \\
&\quad\times\int\frac{\mathcal{D}\pi_N(\mathbf{x})}{2\pi}\int\frac{\mathcal{D}\pi_{N-1}(\mathbf{x})}{2\pi}\cdots\int\frac{\mathcal{D}\pi_0(\mathbf{x})}{2\pi} \\
&\quad\times\Pi_{k=0}^{N-1}\exp\Big[i\Delta t\int d^3x\Big[\pi_k(\mathbf{x})\Big(\frac{\phi_{k+1}(\mathbf{x})-\phi_k(\mathbf{x})}{\Delta t}\Big)-\mathcal{H}(t_k,\mathbf{x})\Big]\Big].
\end{aligned}
\tag{35}
$$

  The initial and final vacuum states in the definition of generating functional (17) are represented by the requirement that $\phi_N(\mathbf{x})=\phi_0(\mathbf{x})=0$. In the continuum limit it corresponds to the condition $\phi(t\to\pm\infty,\mathbf{x})=0$.

- Taking the continuum limit in Eq. (35), we get

$$
W_0[J]=C_0\int\mathcal{D}\phi(t,\mathbf{x})\int\frac{\mathcal{D}\pi(t,\mathbf{x})}{2\pi}\exp\Big[i\int dt\int d^3x\Big(\pi(t,\mathbf{x})\dot{\phi}(t,\mathbf{x})-\mathcal{H}(t,\mathbf{x})\Big)\Big],
\tag{36}
$$

  where the functional integrals are over the configurations of ϕ and π not in space but in space-time. The normalization constant C_0 is chosen in such a way that $W_0[J=0]=1$.

- It is tempting to identify the expression $\big(\pi(t,\mathbf{x})\dot{\phi}(t,\mathbf{x})-\mathcal{H}(t,\mathbf{x})\big)$ with the Lagrangian density. However, we should remember that $\pi(t,\mathbf{x})$ is not a momentum conjugate to $\phi(t,\mathbf{x})$ but an eigenvalue of the momentum operator $\hat{\pi}(t,\mathbf{x})$.

- Since the Hamiltonian (8) is quadratic in conjugate momenta, the integrals over momenta can be easily performed. For this purpose we discretize the space-time using a single index $i=1,\,2,\ldots N$ to numerate the points and $\Delta^{(4)}V$ to denote the four-volume of elementary cube in the discretized Minkowski space. Then,

$$W_0[J] = \lim_{N\to\infty} \int d\phi_1 \int d\phi_2 \cdots \int d\phi_N \int \frac{d\pi_1}{2\pi} \int \frac{d\pi_2}{2\pi} \cdots \int \frac{d\pi_N}{2\pi} \tag{37}$$

$$\times \exp\Big[i\Delta^{(4)}V \sum_i^N \Big(\pi_i\dot{\phi}_i - \frac{1}{2}\pi_i^2 - \frac{1}{2}(\nabla\phi)_i^2 - \frac{1}{2}m^2\phi_i^2 + \phi_i J_i\Big)\Big]$$

$$= \lim_{N\to\infty} \Big(\frac{1}{2\pi i\Delta^{(4)}V}\Big)^{N/2} \int d\phi_1 \int d\phi_2 \cdots \int d\phi_N$$

$$\times \exp\Big[i\Delta^{(4)}V \sum_i^N \Big(\frac{1}{2}\dot{\phi}_i^2 - \frac{1}{2}(\nabla\phi)_i^2 - \frac{1}{2}m^2\phi_i^2 + \phi_i J_i\Big)\Big],$$

where the momentum integrals have been computed according to the formula

$$\int_{-\infty}^{\infty} dp\, e^{ipx-ap^2} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{x^2}{4a}}. \tag{38}$$

- Going to the continuum limit, the expression (37) provides

$$W_0[J] = C_0 \int \mathcal{D}\phi(t,\mathbf{x}) \exp\Big[i\int dt \int d^3x \Big(\pi(t,\mathbf{x})\dot{\phi}(t,\mathbf{x}) - \mathcal{H}(t,\mathbf{x})\Big)\Big], \tag{39}$$

where $\pi(t,\mathbf{x}) \equiv \dot{\phi}(t,\mathbf{x})$ is the classical momentum conjugate to $\phi(t,\mathbf{x})$. Hopefully, denoting two different quantities – momentum eigenvalue and conjugate momentum – with the same symbol $\pi(t,\mathbf{x})$ will not cause a serious confusion.

- The generating functional is finally rewritten as

$$\boxed{W_0[J] = C_0 \int \mathcal{D}\phi(x) \exp\big[i\int d^4x\, \mathcal{L}(x)\big],} \tag{40}$$

where $\mathcal{L}(x) = \mathcal{L}_0(x) + \phi(x)\, J(x)$, see Eq. (1), and the normalization constant, which guarantees that $W_0[J=0] = 1$, equals

$$C_0^{-1} = \int \mathcal{D}\phi(x) \exp\big[i\int d^4x \mathcal{L}_0(x)\big]. \tag{41}$$

Free Green's functions

- The $n-$point Feynman Green function, which is the vacuum expectation value of the chronologically ordered products of the field operators in Heisenberg picture, is defined as

$$i\Delta_0^{(n)}(x_1,\ldots,x_n) = \langle 0|T\hat{\phi}(x_1)\ \ldots\ \hat{\phi}(x_n)|0\rangle, \tag{42}$$

where T is a time ordering operator.

- One observes that the $n-$point Green function is given as the functional derivative of the generating functional (40)

$$\boxed{i\Delta_0^{(n)}(x_1,\ldots,x_n) = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W_0[J]\Big|_{J=0}.} \tag{43}$$

- Computing the derivatives and putting $J = 0$, one indeed finds

$$i\Delta_0^{(n)}(x_1, \ldots, x_n) = C_0 \int \mathcal{D}\phi(x)\, \phi(x_1)\, \phi(x_2) \ldots\, \phi(x_n) \exp\Big[i \int d^4x \mathcal{L}_0(x)\Big]. \tag{44}$$

- To better see the relation of the Green function (42) with the vacuum-vacuum transition amplitude (17) written in the Schrödinger picture, the formula (42) should be rewritten in the Schrödinger picture. For simplicity we consider here only the case of $n = 2$ with $t_2 > t_1$ and we assume that $\hat{H}$ is time independent. Then,

$$i\Delta_0^{(2)}(x_1, x_2) = \langle 0| U(\infty, t_2)\, \phi_{\rm S}(x_2)\, U(t_2, t_1)\, \phi_{\rm S}(x_1) U(t_1, -\infty)|0\rangle, \tag{45}$$

where $U(t_2, t_1) \equiv T \exp\big[-i\int_{t_1}^{t_2} dt\, \hat{H}(t)\big]$. Starting with the expression (45) and following the same steps, which led us from the transition amplitude (17) to the generating functional (40), we obtain the Green function expressed as in Eq. (44).

- We note that the formula (43) can be expressed as the Taylor expansion of the generating functional (40)

$$W_0[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 d^4x_2 \ldots d^4x_n\, i\Delta_0^{(n)}(x_1, \ldots, x_n)\, J(x_1) J(x_2) \ldots J(x_n). \tag{46}$$

Explicit form of free generating functional

- Since the free Lagrangian depends quadratically on the fields, the functional integral in Eq. (40) can be computed explicitly, using the integral formula

$$\int d^n x \exp\Big[-\frac{1}{2}\mathbf{x}\, A\, \mathbf{x} - \mathbf{b}\cdot\mathbf{x}\Big] = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\Big[\frac{1}{2}\mathbf{b}\, A^{-1}\mathbf{b}\Big]. \tag{47}$$

- With the substitutions $A \to i(\partial_\mu\partial^\mu + m^2)$ and $b \to -iJ(x)$, the generating functional (40) gets the following explicit form

$$\boxed{W_0[J] = \exp\Big[-\frac{i}{2}\int d^4x d^4y J(x) \Delta_F(x-y) J(y)\Big],} \tag{48}$$

where the normalization constant is absent as the condition $W_0[J = 0] = 1$ is trivially satisfied; $\Delta_F(x)$ is the Feynman propagator which is an inverse of the operator $\partial_\mu\partial^\mu + m^2$ that is it satisfies the equation

$$[\partial_\mu\partial^\mu + m^2]\Delta_F(x) = -\delta^{(4)}(x). \tag{49}$$

The Feynman boundary condition must be chosen because the functional (48) generates the time-ordered Green's functions.

- As we remember, the Feynman propagator equals

$$\Delta_F(x) = \int \frac{d^4x}{(2\pi)^4}\, e^{-ipx} \Delta_F(x), \qquad \Delta_F(p) = \frac{1}{p^2 - m^2 + i0^+}. \tag{50}$$

Explicit form of free Green's functions

- Using the free generating functional derived in the previous section, we compute here the $n-$point Green's functions starting with the two-point function which is

$$i\Delta_0^{(2)}(x_1,x_2) = -\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\exp\Big[-\frac{i}{2}\int d^4x_1' d^4x_2' J(x_1')\,\Delta_F(x_1'-x_2')\,J(x_2')\Big]\Big|_{J=0}. \quad (51)$$

- The first differentiation leads us to

$$\begin{aligned} i\Delta_0^{(2)}(x_1,x_2) &= \frac{i}{2}\frac{\delta}{\delta J(x_1)}\exp\Big[-\frac{i}{2}\int d^4x_1' d^4x_2' J(x_1')\,\Delta_F(x_1'-x_2')\,J(x_2')\Big] \quad (52)\\ &\times\Big[\int d^4x_2'\Delta_F(x_2-x_2')\,J(x_2') + \int d^4x_1' J(x_1')\,\Delta_F(x_1'-x_2)\Big]\Big|_{J=0}. \end{aligned}$$

The second derivative acts on the product of two functionals. Taking into account that at the end we put $J=0$, one observes that the term coming from the derivative of the first functional vanishes. Thus, one immediately finds

$$\begin{aligned} i\Delta_0^{(2)}(x_1,x_2) &= \frac{i}{2}\frac{\delta}{\delta J(x_1)}\Big[\int d^4x_2'\Delta_F(x_2-x_2')\,J(x_2') + \int d^4x_1' J(x_1')\,\Delta_F(x_1'-x_2)\Big]\Big|_{J=0}\\ &= \frac{i}{2}\big[\Delta_F(x_2-x_1)+\Delta_F(x_1-x_2)\big]. \end{aligned} \quad (53)$$

- Taking into account the symmetry $\Delta_F(x)=\Delta_F(-x)$, we obtain

$$i\Delta_0^{(2)}(x_1,x_2) = i\Delta_F(x_1-x_2). \quad (54)$$

- One observes that $\Delta_0^{(3)}$ and all other free Green's functions of odd n vanish.
- Let us consider the four-point function given as

$$i\Delta_0^{(4)}(x_1,x_2,x_3,x_4) = \Big(\frac{1}{i}\Big)^4\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}W_0[J]\Big|_{J=0}. \quad (55)$$

- The computation is much simplified by observation that this is the third term of Taylor expansion of the exponential function from $W_0[J]$ which determines $\Delta_0^{(4)}$. The first and the second terms are canceled by the differentiation while the fourth and higher vanish after the limit $J=0$ is taken. Thus, one writes

$$\begin{aligned} i\Delta_0^{(4)}(x_1,x_2,x_3,x_4) &= \frac{1}{2!}\Big(\frac{i}{2}\Big)^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)} \quad (56)\\ &\times\int d^4x_1' d^4x_2' d^4x_3' d^4x_4' J(x_1')\,\Delta_F(x_1'-x_2')\,J(x_2')\,J(x_3')\,\Delta_F(x_3'-x_4')\,J(x_4')\Big|_{J=0}. \end{aligned}$$

- The differentiation becomes simpler when the formula (56) is written symbolically as

$$i\Delta_0^{(4)}(1,2,3,4) = \frac{1}{2!}\Big(\frac{i}{2}\Big)^2\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}\sum_{i,j,k,l} J_iJ_jJ_kJ_l\Delta_F^{ij}\Delta_F^{kl}\Big|_{J=0}, \quad (57)$$

where $J_m \equiv J(x_m)$ with $m=1,2,3,4$, and $J_m \equiv J(x_m')$ with $m=i,j,k,l$; the sums represent the integrals.

- After computing all four derivatives, one finds

$$i\Delta_0^{(4)}(1,2,3,4) = \frac{1}{2!}\Big(\frac{i}{2}\Big)^2 \sum_{i,j,k,l} \tag{58}$$

$$\big(\delta_{i4}\delta_{j3}\delta_{k2}\delta_{l1} + \delta_{i4}\delta_{j3}\delta_{k1}\delta_{l2} + \delta_{i4}\delta_{j2}\delta_{k3}\delta_{l1} + \delta_{i4}\delta_{j1}\delta_{k3}\delta_{l2} + \delta_{i4}\delta_{j2}\delta_{k1}\delta_{l3} + \delta_{i4}\delta_{j1}\delta_{k2}\delta_{l3}$$
$$+ \delta_{i3}\delta_{j4}\delta_{k2}\delta_{l1} + \delta_{i3}\delta_{j4}\delta_{k1}\delta_{l2} + \delta_{i2}\delta_{j4}\delta_{k3}\delta_{l1} + \delta_{i1}\delta_{j4}\delta_{k3}\delta_{l2} + \delta_{i2}\delta_{j4}\delta_{k1}\delta_{l3} + \delta_{i1}\delta_{j4}\delta_{k2}\delta_{l3}$$
$$+ \delta_{i3}\delta_{j2}\delta_{k4}\delta_{l1} + \delta_{i3}\delta_{j1}\delta_{k4}\delta_{l2} + \delta_{i2}\delta_{j3}\delta_{k4}\delta_{l1} + \delta_{i1}\delta_{j3}\delta_{k4}\delta_{l2} + \delta_{i2}\delta_{j1}\delta_{k4}\delta_{l3} + \delta_{i1}\delta_{j2}\delta_{k4}\delta_{l3}$$
$$+ \delta_{i3}\delta_{j2}\delta_{k1}\delta_{l4} + \delta_{i3}\delta_{j1}\delta_{k2}\delta_{l4} + \delta_{i2}\delta_{j3}\delta_{k1}\delta_{l4} + \delta_{i1}\delta_{j3}\delta_{k2}\delta_{l4} + \delta_{i2}\delta_{j1}\delta_{k3}\delta_{l4} + \delta_{i1}\delta_{j2}\delta_{k3}\delta_{l4}\big)\Delta_F^{ij}\Delta_F^{kl}.$$

- Because of the symmetry $\Delta_F^{ij} = \Delta_F^{ji}$, the number of terms is dramatically reduced. Performing the reduction step by step, one obtains

$$\begin{aligned} i\Delta_0^{(4)}(1,2,3,4) &= -\sum_{i,j,k,l}\big(\delta_{i4}\delta_{j3}\delta_{k2}\delta_{l1} + \delta_{i4}\delta_{j2}\delta_{k3}\delta_{l1} + \delta_{i4}\delta_{j1}\delta_{k3}\delta_{l2}\big)\Delta_F^{ij}\Delta_F^{kl} \\ &= -\Delta_F^{34}\Delta_F^{12} - \Delta_F^{24}\Delta_F^{13} - \Delta_F^{14}\Delta_F^{23}. \end{aligned} \tag{59}$$

which is finally written as

$$\begin{aligned} i\Delta_0^{(4)}(x_1,x_2,x_3,x_4) = -\Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) - \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) \\ -\Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3). \end{aligned} \tag{60}$$

- The result (60) is actually an example of the Wick's theorem, to be discussed further on, which, in particular, states that any $n-$point free Green function can be written as a sum over all possible products of $n/2$ two-point Green functions $\Delta_F(x_i - x_j)$.

Self-interacting Scalar Field

- We include now an interaction into consideration. Specifically, we study here the self-interacting scalar field with the Lagrangian density given as

$$\mathcal{L}(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2(x)\phi^2(x) - \frac{\lambda}{4!}\phi^4(x), \tag{61}$$

where λ is the coupling constant.

- The field obeys the equation of motion

$$\big[\partial^2 + m^2\big]\phi(x) = -\frac{\lambda}{3!}\phi^3(x). \tag{62}$$

- The generating functional is *not* derived but *postulated* in the expected form

$$W[J] = C\int \mathcal{D}\phi(x)\exp\Big\{i\Big[S_0 + S_I + \int d^4x\, J(x)\phi(x)\Big]\Big\}, \tag{63}$$

where C is the normalization constant discussed later on, the free action is given by Eq. (4) and

$$S_I[\phi] = \int d^4x\mathcal{L}_I(x) = -\frac{\lambda}{4!}\int d^4x\,\phi^4(x). \tag{64}$$

- Making use of a simple observation that

$$\exp(\alpha x^2 + \beta x^4 + jx) = \exp\Big(\beta \frac{d^4}{dj^4}\Big) \exp(\alpha x^2 + jx), \tag{65}$$

the generating functional (63) can be written as

$$\boxed{W[J] = C \exp\Big\{ i S_I \Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big\} W_0[J],} \tag{66}$$

where $W_0[J]$ is the generating functional of free fields (48) and

$$S_I\Big[\frac{1}{i}\frac{\delta}{\delta j}\Big] = -\frac{\lambda}{4!}\Big(\frac{1}{i}\Big)^4 \int d^4x \frac{\delta^4}{\delta j^4(x)} = -\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j^4(x)}. \tag{67}$$

- The generating functional of the form (66) is used to obtain a perturbative expansion of any Green's function or, equivalently, to derive the Feynman rules of the expansion.

Two-point Green's function

- To derive the two-point Green's function of interacting scalar fields we have to compute the following expression

$$i\Delta^{(2)}(x_1, x_2) = \Big(\frac{1}{i}\Big)^2 \frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)} \exp\Big\{ - i\frac{\lambda}{4!}\Big(\frac{1}{i}\Big)^4 \int d^4x \frac{\delta^4}{\delta J^4(x)}\Big\} W_0[J]\Big|_{J=0}. \tag{68}$$

- Assuming temporarily that the normalization constant C in Eq. (66) equals unity, one finds

$$\begin{aligned} i\Delta^{(2)}(x_1, x_2) &= \Big(\frac{1}{i}\Big)^2 \frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)} \\ &\times \exp\Big\{ - i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J^4(x)}\Big\} \exp\Big\{ -\frac{i}{2}\int d^4x_1' d^4x_2' J(x_1')\Delta_F(x_1' - x_2') J(x_2')\Big\}\Big|_{J=0} \end{aligned} \tag{69}$$

where we have used the explicit form of the free generating functional given by Eq. (48).

- Expanding the two exponentials in Eq. (69) into Taylor series, we get

$$\begin{aligned} i\Delta^{(2)}(x_1, x_2) &= -\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)} \\ &\times\Big\{1 - \frac{i\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J^4(x)} + \frac{1}{2!}\Big(\frac{-i\lambda}{4!}\Big)^2\Big[\int d^4x \frac{\delta^4}{\delta J^4(x)}\Big]^2 + \cdots\Big\} \\ &\times\Big\{1 - \frac{i}{2}\int d^4x_1' d^4x_2' J(x_1')\Delta_F(x_1' - x_2') J(x_2') \\ &+ \frac{1}{2!}\Big(\frac{i}{2}\Big)^2 \int d^4x_1' d^4x_2' d^4x_1'' d^4x_2'' J(x_1')\Delta_F(x_1' - x_2') J(x_2') J(x_1'')\Delta_F(x_1'' - x_2'') J(x_2'') + \cdots\Big\}\Big|_{J=0}. \end{aligned} \tag{70}$$

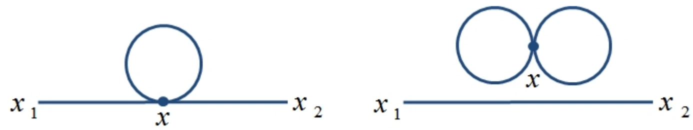

Figure 1: Diagrammatic representation of the first order contributions to the two-point Green's function.

Order λ

- One observes that the only terms that survive are those where the number of derivatives with respect to J equals the number J's. The contribution of the order λ, which is labeled with the index '1', is written symbolically as

$$i\Delta_1^{(2)}(1,2) = -\frac{i\lambda}{3!\,4!}\left(\frac{i}{2}\right)^3 \frac{\partial}{\partial J_1}\frac{\partial}{\partial J_2}\sum_a \frac{\delta^4}{\delta J_a^4}\sum_{ijklmn} J_iJ_jJ_kJ_lJ_mJ_n\Delta_F^{ij}\Delta_F^{kl}\Delta_F^{mn}, \quad (71)$$

  where the sums represent the integrals.

- We have here three categories of space-time points: the external points x_1, x_2 represented by 1 and 2, the interaction point x represented by the index 'a' and the intermediate points x_1', x_2', x_1'', x_2'' represented by the indices i, j, k, l which disappear in the final expression. We note that it is no longer needed to set $J = 0$.

- After taking the derivative of the order 6 in Eq. (71), one gets $6! = 720$ terms of two generic forms

$$\sum_a \sum_{ijklmn} \delta^{i1}\delta^{j2}\delta^{ka}\delta^{la}\delta^{ma}\delta^{na}\Delta_F^{ij}\Delta_F^{kl}\Delta_F^{mn} = \sum_a \Delta_F^{12}\Delta_F^{aa}\Delta_F^{aa}, \quad (72)$$

$$\sum_a \sum_{ijklmn} \delta^{i1}\delta^{j2}\delta^{ka}\delta^{la}\delta^{ma}\delta^{na}\Delta_F^{ij}\Delta_F^{kl}\Delta_F^{mn} = \sum_a \Delta_F^{1a}\Delta_F^{2a}\Delta_F^{aa}. \quad (73)$$

  Let us compute the number of terms keeping in mind the symmetry $\Delta_F^{ij} = \Delta_F^{ji}$. In case of the first type there are six possibilities to assign the index '1' to the product of three Green's functions and only one possibility to assign the index '2'. There are 4! possibilities to assign four indices 'a' to the Green's functions. Thus, there are $6 \cdot 4! = 144$ terms of the first type.

- To compute the number of terms of the second type, one observes that there are six possibilities to assign the index '1' and four possibilities to assign the index '2'. There are again 4! possibilities to assign four indices 'a'. Consequently, there are $6 \cdot 4 \cdot 4! = 576$ terms of the second type. So, there are $144 + 576 = 720$ terms.

- Taking into account the symmetry $\Delta_F^{ij} = \Delta_F^{ji}$ and computing the numerical coefficient from Eq. (71) as $3! \cdot 4! \cdot 2^3 = 1152$, one finds

$$i\Delta_1^{(2)}(x_1,x_2) = -\frac{\lambda}{2}\int d^4x\; \Delta_F(x_1-x)\Delta_F(x-x)\Delta_F(x-x_2) - \frac{\lambda}{8}\Delta_F(x_1-x_2)\int d^4x\;\big(\Delta_F(x-x)\big)^2, \quad (74)$$

  which is represented by the diagrams in Fig. 1.

- One observes that the sum of zeroth and first contributions can be written as the following product

$$i\Delta^{(2)}(x_1,x_2) = \Big(i\Delta_F(x_1-x_2) - \frac{\lambda}{2}\int d^4x\ \Delta_F(x_1-x)\Delta_F(x-x)\Delta_F(x-x_2)\Big) \times \Big(1 + i\frac{\lambda}{8}\int d^4x\ \big(\Delta_F(x-x)\big)^2\Big) + \mathcal{O}(\lambda^2). \tag{75}$$

- The second bracket includes, except the unity, the so-called *disconnected* diagram from Fig. 1 which represents the vacuum-vacuum transition amplitude of the order λ. The first bracket gives the so-called *connected* Green's function which is of our actual interest.

- The disconnected part of the Green's functions representing the vacuum-vacuum transition amplitude can be excluded systematically by choosing the normalization constant C in the definition of generating functional (66) in such a way that $C^{-1} = W[J=0]$ which just equals the vacuum-vacuum transition amplitude.

- The normalization constant is

$$\begin{aligned} C^{-1} &= \exp\Big\{iS_I\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big\}W_0[J]\Big|_{J=0} \qquad (76)\\ &= \exp\Big\{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta J^4(x)}\Big\}\exp\Big\{-\frac{i}{2}\int d^4x_1'd^4x_2'J(x_1')\Delta_F(x_1'-x_2')J(x_2')\Big\}\Big|_{J=0}. \end{aligned}$$

Expanding C^{-1} up to $\mathcal{O}(\lambda^2)$ and using the symbolic notation, we get

$$\begin{aligned} C^{-1} &= 1 + \Big(-\frac{i\lambda}{4!}\Big)\frac{1}{2!}\Big(\frac{i}{2}\Big)^2\sum_a\frac{\delta^4}{\delta J_a^4}\sum_{ijkl}J_iJ_jJ_kJ_l\Delta_F^{ij}\Delta_F^{kl} \qquad (77)\\ &= 1 + \frac{i\lambda}{192}\sum_a\frac{\delta^4}{\delta J_a^4}\sum_{ijkl}J_iJ_jJ_kJ_l\Delta_F^{ij}\Delta_F^{kl} = 1 + \frac{i\lambda}{192}4!\sum_a\Delta_F^{aa}\Delta_F^{aa} = 1 + \frac{i\lambda}{8}\sum_a\Delta_F^{aa}\Delta_F^{aa}, \end{aligned}$$

which in the standard notation coincides with the second bracket in Eq. (75).

- The connected Green's function (labeled with the index 'c') of the order λ thus equals

$$i\Delta_c^{(2)}(x_1,x_2) = i\Delta_F(x_1-x_2) - \frac{\lambda}{2}\int d^4x\ \Delta_F(x_1-x)\Delta_F(x-x)\Delta_F(x-x_2). \tag{78}$$

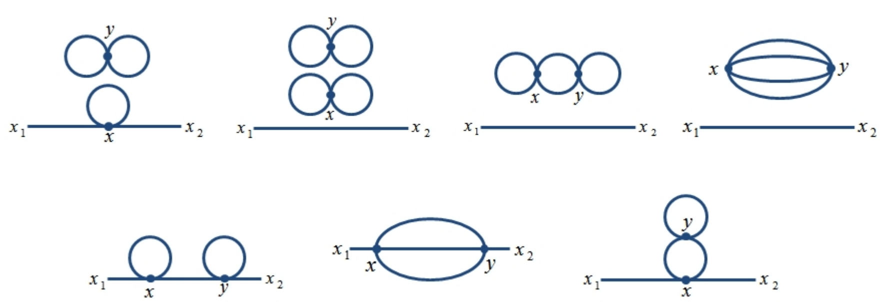

Figure 2: Diagrammatic representation of the second order contributions to the two-point Green's function.

Figure 3: Diagrammatic representation of the complete two-point Green's function.

Order λ^2

- The analysis of the order λ^2 is much more tedious but the result can be easily guessed.
- If the normalization constant C is really a constant, the diagrams corresponding to the second order contributions to the two-point Green's function are shown in Fig. 2.
- One observes that the two-point Green's function, which includes the contributions up to the second order, can be expressed as the product represented graphically in Fig. 3.
- The second order contributions to the normalization constant given by Eq. (76) are represented by the diagrams shown in Fig. 4.
- One sees that combining the zeroth, first and second order contributions to the normalization constant (76), one gets the diagrams from the right bracket in Fig. 3.
- If we include normalization constant (76) in the generating functional (66), the disconnected subgraphs, corresponding to the vacuum-vacuum transition amplitude, are eliminated and the connected two-point Green's function is of the form diagrammatically represented by the left bracket of Fig. 3.

Generating functional of connected diagrams

- Instead of modifying the normalization constant, one can define the new generating functional

$$Z[J] = -i \ln W[J], \tag{79}$$

which independently of the normalization constant generates the connected Green's functions as

$$\boxed{\Delta_c^{(n)}(x_1, \ldots, x_n) = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\bigg|_{J=0}.} \tag{80}$$

- In case of non-interacting fields, the explicit form of the generating functional $Z[J]$ is

$$Z_0[J] = -\frac{1}{2} \int d^4x d^4y J(x) \Delta_F(x-y) J(y), \tag{81}$$

where the formula (48) has been used.

Figure 4: Diagrammatic representation of the second order contributions to C^{-1}.

- Eq. (81) immediately shows that

$$\Delta_{0c}^{(n)}(x_1,\ldots,x_n) = (-i)^n \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z_0[J]\Big|_{J=0} = \begin{cases} \Delta_F(x_1-x_2) & \text{if } \; n=2, \\ 0 & \text{if } \; n\neq 2. \end{cases} \tag{82}$$

that is there is no free connected Green's function except the two-point function.

- In case of interacting field, the explicit form of the generating functional of connected diagrams is

$$Z[J] = -i\ln\Big\{\exp\Big(iS_I\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big)\exp\Big[-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\Big]\Big\}, \tag{83}$$

but this form is not very useful.

- Let us see how the formula (80) works for one-, two- and three-point Green's functions. In case of one-point function, we have

$$\Delta_c^{(1)}(x) = -i\frac{\delta Z[J]}{\delta J(x)}\Big|_{J=0} = -\frac{1}{W[J]}\frac{\delta W[J]}{\delta J(x)}\Big|_{J=0} = -i\langle\phi(x)\rangle, \tag{84}$$

where

$$\langle\ldots\rangle \equiv \frac{\int\mathcal{D}\phi(x)\,\ldots\,\exp\big(iS[\phi]\big)}{\int\mathcal{D}\phi(x)\exp\big(iS[\phi]\big)}. \tag{85}$$

So, the one-point function represents the field expectation value.

- In case of two-point Green's function, one finds

$$\begin{aligned} \Delta_c^{(2)}(x_1,x_2) &= -\frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)}\Big|_{J=0} \\ &= i\frac{1}{W[J]}\frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\Big|_{J=0} - i\frac{1}{W[J]}\frac{\delta W[J]}{\delta J(x_1)}\frac{1}{W[J]}\frac{\delta W[J]}{\delta J(x_2)}\Big|_{J=0} \\ &= -i\langle\phi(x_1)\phi(x_2)\rangle + i\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle \\ &= -i\Big\langle\big(\phi(x_1)-\langle\phi(x_1)\rangle\big)\big(\phi(x_2)-\langle\phi(x_2)\rangle\big)\Big\rangle. \end{aligned} \tag{86}$$

- If $\langle\phi(x)\rangle = 0$, as is the case of the Lagrangian (61), the second term of Eq. (86) vanishes while the first term represents the connected Green's function that would be obtained from the generating functional (66) with the normalization constant (76).

- If $\langle\phi(x)\rangle \neq 0$, as is the case of the Lagrangian (61) with $m^2 < 0$, the formula (86) tells us that the connected Green's function should be computed not as the expectation value of the product of fields $\phi(x)$ but as the expectation value of the product of $\phi(x) - \langle\phi(x)\rangle$.

- The computation of the three-point function confirms our observations made with the one-and two-point functions. Indeed,

$$\Delta_c^{(3)}(x_1, x_2, x_3) = i \frac{\delta^3 Z[J]}{\delta J(x_1)\, \delta J(x_2)\, \delta J(x_3)}\bigg|_{J=0} \tag{87}$$

$$= -\frac{1}{W[J]} \frac{\delta^2 W[J]}{\delta J(x_1)\, \delta J(x_2)\, \delta J(x_3)}\bigg|_{J=0} + \frac{2}{W^3[J]} \frac{\delta W[J]}{\delta J(x_1)} \frac{\delta W[J]}{\delta J(x_2)} \frac{\delta W[J]}{\delta J(x_3)}\bigg|_{J=0}$$

$$- \frac{1}{W^2[J]} \left(\frac{\delta^2 W[J]}{\delta J(x_1)\, \delta J(x_2)} \frac{\delta W[J]}{\delta J(x_3)} + \frac{\delta^2 W[J]}{\delta J(x_3)\, \delta J(x_1)} \frac{\delta W[J]}{\delta J(x_2)} + \frac{\delta^2 W[J]}{\delta J(x_2)\, \delta J(x_3)} \frac{\delta W[J]}{\delta J(x_1)} \right)\bigg|_{J=0}$$

$$= i\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle - 2i\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle\langle\phi(x_3)\rangle$$

$$+ i\langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\rangle + i\langle\phi(x_3)\phi(x_1)\rangle\langle\phi(x_2)\rangle + i\langle\phi(x_2)\phi(x_3)\rangle\langle\phi(x_1)\rangle$$

$$= i\Big\langle \big(\phi(x_1) - \langle\phi(x_1)\rangle\big)\big(\phi(x_2) - \langle\phi(x_2)\rangle\big)\big(\phi(x_3) - \langle\phi(x_3)\rangle\big) \Big\rangle.$$

- We close the lecture with the Taylor expansion of the generating functional (79) which reads

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 d^4x_2 \ldots d^4x_n\, \Delta_c^{(n)}(x_1, \ldots, x_n)\, J(x_1)J(x_2)\ldots J(x_n). \tag{88}$$