Functional Methods in Statistical QFT

In this Lecture we discuss the generating functionals of Matrsubara imaginary-time formalism and of the Keldysh-Schwinger real-time formalism. The discussion is largely based on the analysis of the generating functional in vacuum QFT presented in Lecture X.

Preliminaries

- We consider again the real scalar field ϕ. At the beginning the field is assumed to interact only with an external source J. The self-interacting field is discussed further on. The classical Lagrangian density is

$$\mathcal{L} = \mathcal{L}_0 + J(x)\phi(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) + J(x)\phi(x). \tag{1}$$

- The equation of motion is

$$\left[\partial_\mu\partial^\mu + m^2\right]\phi(x) = J(x). \tag{2}$$

- The action of free field equals

$$S_0[\phi] = \frac{1}{2}\int d^4x\mathcal{L}_0 = \int d^4x(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2). \tag{3}$$

- The Hamiltonian density and the conjugate momentum are defined as

$$\mathcal{H}(x) = \pi\dot{\phi}(x) - \mathcal{L}(x), \qquad \pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\phi}(x)} = \dot{\phi}(x). \tag{4}$$

- The Hamiltonian, the Hamiltonian densities of non-interacting field and of the field interacting with external source $J(x)$ are

$$H(t) = \int d^3x\,\mathcal{H}(x), \tag{5}$$

$$\mathcal{H}_0(x) = \frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) \tag{6}$$

$$\mathcal{H}(x) = \mathcal{H}_0(x) - \phi(x)\,J(x). \tag{7}$$

- When the generating functional of interacting field is studied, the Lagrangian density includes the quartic term $-\frac{\lambda}{4!}\phi^4(x)$.

Generating functional of Matsubara formalism

Free generating functional of Matsubara formalism

- As we remember, the free generating functional in vacuum field theory is defined as

$$W_0[J] = \mathcal{N}\,\langle 0|T\exp\Big[-i\int d^4x\big(\hat{\mathcal{H}}_0(x) - \hat{\phi}(x)\,J(x)\big)\Big]|0\rangle, \tag{8}$$

where $|0\rangle$ denotes the vacuum state, T is the operator of time ordering and the normalization constant $\mathcal{N}$ is chosen in such a way that $W_0[J=0]=1$ which means

$$\frac{1}{\mathcal{N}} = \langle 0|T\exp\Big[-i\int d^4x\,\hat{\mathcal{H}}_0(x)\Big]|0\rangle. \tag{9}$$

The equations (8, 9) are written in the Schrödinger picture.

- The path integral representation of the free generating functional (8), which is derived in Lecture X, is

$$W_0[J] = \mathcal{N} \int \mathcal{D}\phi(x) \exp\Big[i \int d^4x \big(\mathcal{L}_0(x) + \phi(x)\, J(x)\big)\Big], \tag{10}$$

and the normalization constant $\mathcal{N}$ equals

$$\frac{1}{\mathcal{N}} = \int \mathcal{D}\phi(x) \exp\Big[i \int d^4x\, \mathcal{L}_0(x)\Big]. \tag{11}$$

The field $\phi(x)$ vanishes at $t = -\infty$ and $t = \infty$ which reflects the fact that the generating functional (8) is defined as the vacuum expectation value.

- The free generating functional of the Matsubara formalism in the operator formalism is

$$W_0[J] = Z_0^{-1}\, \mathrm{Tr}\Big[\mathcal{T} \exp\Big[-\int_0^\beta d^4x \big(\hat{\mathcal{H}}_0(x) - \hat{\phi}(x)\, J(x)\big)\Big]\Big], \tag{12}$$

where the trace is taken over a complete set of states, $\mathcal{T}$ is the operator of ordering in imaginary time, the position four- vector x should be understood as $x = (-i\tau, \mathbf{x})$ and we use the notation

$$\int_0^\beta d^4x \ldots \equiv \int_0^\beta d\tau \int d^3x \ldots \tag{13}$$

- The normalization constant is the inverse of the partition function defined as

$$Z_0 = \mathrm{Tr}\Big[\mathcal{T} \exp\Big[-\int_0^\beta d^4x\, \hat{\mathcal{H}}_0(x)\Big]\Big]. \tag{14}$$

- The path integral representation of the Matsubara generating functional (12) can be found following the same steps, discussed in detail in Lecture X, which led us to the formula (15). Thus, one finds

$$W_0[J] = Z_0^{-1} \int \mathcal{D}\phi(x) \exp\Big[\int_0^\beta d^4x \big(\mathcal{L}_0(x) + \phi(x)\, J(x)\big)\Big], \tag{15}$$

and

$$Z_0 = \int \mathcal{D}\phi(x) \exp\Big[\int_0^\beta d^4x\, \mathcal{L}_0(x)\Big], \tag{16}$$

where the field $\phi(x)$ obeys the periodic boundary condition that is $\phi(0, \mathbf{x}) = \phi(-i\beta, \mathbf{x})$.

- The field has the same value at $\tau = 0$ and at $\tau = \beta$ because of the trace in the definition (12). In case of generating function of vacuum QFT, the field vanishes at $t = -\infty$ and $t = \infty$ as we then deal with the vacuum expectation value.

- Since the time component of the position four-vector x is $-i\tau$, there is not the action S_0 but rather the so-called Euclidean action S_0^E in Eqs. (15, 16). While the action is

$$S_0[\phi] \equiv \int d^4x\, \mathcal{L}_0 = \frac{1}{2} \int d^4x (\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) = \frac{1}{2} \int d^4x \Big[\Big(\frac{\partial\phi}{\partial t}\Big)^2 - (\nabla\phi)^2 - m^2\phi^2\Big], \tag{17}$$

the Euclidean action equals

$$S_0^E[\phi] \equiv \int_0^\beta d^4x\, \mathcal{L}_0 = \frac{1}{2} \int_0^\beta d^4x \Big[-\Big(\frac{\partial\phi}{\partial\tau}\Big)^2 - (\nabla\phi)^2 - m^2\phi^2\Big]. \tag{18}$$

- The formulas (15, 16) are often written as

$$W_0[J] = Z_0^{-1} \int \mathcal{D}\phi \, \exp\Big[S_0^E[\phi] + \int_0^\beta d^4x \, \phi(x)\, J(x)\Big], \tag{19}$$

and

$$Z_0 = \int \mathcal{D}\phi \, e^{S_0^E[\phi]}. \tag{20}$$

Explicit form of free Matsubara generating functional

- As in case of vacuum generating functional, we can get the explicit form of the free Matsubara generating functional performing the functional Gaussian integral. The result is

$$\boxed{W_0[J] = \exp\Big[-\frac{1}{2}\int_0^\beta d^4x_1 \int_0^\beta d^4x_2 \, J(x_1)\Delta_0(x_1 - x_2)J(x_2)\Big],} \tag{21}$$

where $x_1 = (-i\tau_1, \mathbf{x}_1)$, $x_2 = (-i\tau_2, \mathbf{x}_2)$ and the free temperature Green's function $\Delta_0(x)$ satisfies the equation of motion

$$\Big(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\Big)\Delta_0(x) = -\delta^{(4)}(x). \tag{22}$$

Generating functional of self-interacting field

- Now, the field self-interaction is taken into consideration by including the quartic term $-\frac{\lambda}{4!}\phi^4(x)$ in the Lagrangian density. The generating functional becomes

$$W[J] = Z^{-1} \int \mathcal{D}\phi(x) \exp\Big[S_0^E + S_I^E + \int_0^\beta d^4x \, J(x)\phi(x)\Big], \tag{23}$$

where

$$S_I^E[\phi] = \int_0^\beta d\tau \int d^3x \, \mathcal{L}_I(x) = -\frac{\lambda}{4!}\int_0^\beta d^4x \, \phi^4(x), \tag{24}$$

and the partition function Z is

$$Z = \int \mathcal{D}\phi(x) \exp\big[S_0^E + S_I^E\big]. \tag{25}$$

- As in case of vacuum QFT, one makes use of the observation that

$$\exp(\alpha x^2 + \beta x^4 + jx) = \exp\Big(\beta \frac{d^4}{dj^4}\Big) \exp(\alpha x^2 + jx), \tag{26}$$

and rewrites the generating functional (23) as

$$\boxed{W[J] = Z^{-1} \exp\Big\{S_I^E\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big\} W_0[J],} \tag{27}$$

where $W_0[J]$ is the generating functional of free fields (21),

$$S_I^E\Big[\frac{\delta}{\delta J}\Big] = -\frac{\lambda}{4!}\int_0^\beta d^4x \, \frac{\delta^4}{\delta J^4(x)}, \tag{28}$$

and the partition function equals

$$Z = \exp\Big\{S_I^E\Big[\frac{\delta}{\delta J}\Big]\Big\} W_0[J]\Big|_{J=0}. \tag{29}$$

- One observes that the generating functional (27) satisfies the normalization condition $W[J=0]=1$.
- The generating functional (27) can be used to obtain a perturbative expansion of $n-$point temperature Green's function. The computation is fully analogous to that in vacuum QFT.

Generating functional of Keldysh-Schwinger formalism

Free generating functional

- As we remember, the contour Green's function of scalar field is defined as

$$i\Delta(x_1,x_2) = Z^{-1}\mathrm{Tr}\big[\hat{\rho}(t_0)\tilde{T}\hat{\phi}(x_1)\hat{\phi}(x_2)\big], \tag{30}$$

where $Z \equiv \mathrm{Tr}[\hat{\rho}(t_0)]$.

- The density operator $\hat{\rho}(t_0)$ can be expressed through the eigenstates of the field operator $\hat{\phi}(t_0 = -\infty \pm i0^+, \mathbf{x})$ as

$$\hat{\rho} = \int D\phi'(\mathbf{x})\, D\phi''(\mathbf{x})\rho\big[\phi'(\mathbf{x}),\phi''(\mathbf{x})\big]\, |\phi''(\mathbf{x})\rangle\langle\phi'(\mathbf{x})|, \tag{31}$$

where

$$\phi'(\mathbf{x}) \equiv \phi(t=-\infty+i0^+,\mathbf{x}), \qquad\qquad \phi''(\mathbf{x}) \equiv \phi(t=-\infty-i0^+,\mathbf{x}). \tag{32}$$

- The times $-\infty + i0^+$ and $-\infty - i0^+$ correspond to the beginning of the upper branch and the end of the lower branch, respectively, of the Keldysh contour shown in Fig. 1.
- To derive the generating functional, we first calculate Z which is

$$\begin{aligned} Z = \mathrm{Tr}[\hat{\rho}] \;&=\; \int D\phi_{-\infty}(\mathbf{x})\langle\phi_{-\infty}(\mathbf{x})|\hat{\rho}|\phi_{-\infty}(\mathbf{x})\rangle \\ &=\; \int D\phi_{-\infty}(\mathbf{x})\, D\phi'(\mathbf{x})\, D\phi''(\mathbf{x})\, \rho\big[\phi'(\mathbf{x})|\phi''(\mathbf{x})\big]\, \langle\phi_{-\infty}(\mathbf{x})|\phi''(\mathbf{x})\rangle\langle\phi'(\mathbf{x})|\phi_{-\infty}(\mathbf{x})\rangle, \end{aligned} \tag{33}$$

where

$$\phi_{-\infty}(\mathbf{x}) \equiv \phi(t=-\infty,\mathbf{x}). \tag{34}$$

- Since the eigenstates of the field operator are assumed to be mutually orthogonal and

$$\langle\phi_{-\infty}|\phi''\rangle = \delta[\phi_{-\infty}-\phi''], \qquad\qquad \langle\phi'|\phi_{-\infty}\rangle = \delta[\phi''-\phi_{-\infty}], \tag{35}$$

the formula (33) gets the expected form

$$Z = \int D\phi_{-\infty}\, D\phi'\, D\phi''\, \rho[\phi'|\phi'']\, \delta[\phi''-\phi_{-\infty}]\, \delta[\phi_{-\infty}-\phi'] = \int D\phi\, \rho[\phi|\phi]. \tag{36}$$

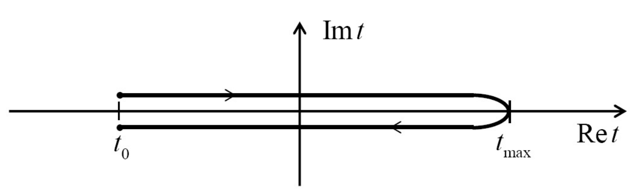

Figure 1: The Keldysh contour

- The contour Green's function can be written as

$$i\Delta(x_1,x_2) = Z^{-1}\int D\phi''(\mathbf{x})\, D\phi'(\mathbf{x})\, \rho\big[\phi'(\mathbf{x})\big|\phi''(\mathbf{x})\big]\big\langle \phi''(\mathbf{x})\big|\tilde{T}\hat{\phi}(x_1)\hat{\phi}(x_2)\big|\phi'(\mathbf{x})\big\rangle. \tag{37}$$

- Repeating the steps which led us to the the path-integral representation of the Feynman propagator of the vacuum QFT, one finds the path-integral representation of the propagator

$$\langle \phi''(\mathbf{x})|\tilde{T}\hat{\phi}(x_1)\hat{\phi}(x_2)|\phi'(\mathbf{x})\rangle = \int_{\substack{\phi(-\infty+i0^+,\mathbf{x})=\phi'(\mathbf{x})\\ \phi(-\infty-i0^+,\mathbf{x})=\phi''(\mathbf{x})}} \mathcal{D}\phi(x)\, \phi(x_1)\, \phi(x_2)\, \exp\Big[i\int_C d^4x\, \mathcal{L}(x)\Big], \tag{38}$$

where the functional integral is performed over the field configurations in time and space with the boundary condition (32). The time integration over the Lagrangian density is now performed along the Keldysh contour.

- Inserting the propagator (38) into Eq. (37), the Green's function equals

$$\begin{aligned} i\Delta(x_1,x_2) &= Z^{-1}\int D\phi'(\mathbf{x})\, D\phi''(\mathbf{x})\, \rho\big[\phi'(\mathbf{x})\big|\phi''(\mathbf{x})\big] \qquad (39) \\ &\quad \times \int_{\substack{\phi(-\infty+i0^+,\mathbf{x})=\phi'(\mathbf{x})\\ \phi(-\infty-i0^+,\mathbf{x})=\phi''(\mathbf{x})}} \mathcal{D}\phi(x)\, \phi(x_1)\, \phi(x_2)\, \exp\Big[i\int_C d^4x \mathcal{L}(x)\Big]. \end{aligned}$$

- The formula (39) suggests the form of the generating functional

$$\begin{aligned} W_0[J] &= \mathcal{N}\int D\phi'(\mathbf{x})\, D\phi''(\mathbf{x})\, \rho\big[\phi'(\mathbf{x})\big|\phi''(\mathbf{x})\big] \qquad (40) \\ &\quad \times \int_{\substack{\phi(-\infty+i0^+,\mathbf{x})=\phi'(\mathbf{x})\\ \phi(-\infty-i0^+,\mathbf{x})=\phi''(\mathbf{x})}} \mathcal{D}\phi(x)\, \exp\Big[i\int_C d^4x\Big(\mathcal{L}_0(x) + \phi(x)\, J(x)\Big)\Big], \end{aligned}$$

where instead of the Lagrangian density (1) the free Lagrangian $\mathcal{L}_0$ is used to show the dependence of the generating functional on the current $J(x)$.

- Let us observe that the expression in the second line of Eq. (40) is like the generating function of vacuum QFT with the modified boundary conditions – the fields do not vanish at the boundary but are finite. The functional (40) can be written as

$$W_0[J] = \int D\phi'(\mathbf{x})\, D\phi''(\mathbf{x})\, \rho\big[\phi'(\mathbf{x})\big|\phi''(\mathbf{x})\big]\; W_0^{\rm vac}[\phi'(\mathbf{x}),\phi''(\mathbf{x}),J], \tag{41}$$

where

$$W_0^{\rm vac}[\phi'(\mathbf{x}),\phi''(\mathbf{x}),J] \equiv \int_{\substack{\phi(-\infty+i0^+,\mathbf{x})=\phi'(\mathbf{x})\\ \phi(-\infty-i0^+,\mathbf{x})=\phi''(\mathbf{x})}} \mathcal{D}\phi(x)\, \exp\Big[i\int_C d^4x\Big(\mathcal{L}_0(x) + \phi(x)\, J(x)\Big)\Big]. \tag{42}$$

- The $n-$point contour Green's function is generated as

$$i\Delta(x_1,x_2,\dots x_n) = \frac{1}{W_0[J=0]}\,\frac{1}{i^n}\,\frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}W_0[J]\bigg|_{J=0}. \tag{43}$$

- Computing the derivatives with respect to the current, one should remember that

$$\frac{\delta J(x)}{\delta J(y)} = \delta_C^{(4)}(x,y), \tag{44}$$

because x_0 and y_0 are on the Keldysh contour.

Example: generation of $\Delta^>(x,y)$

- To see how the formula (43) works, let us derive the function $\Delta^>(x_1,x_2)$ which according Eq. (43) is
$$i\Delta^>(x,y) = \frac{1}{W[J=0]}\frac{1}{i^2}\frac{\delta}{\delta J_+(x)}\frac{\delta}{\delta J_-(y)}W[J]\bigg|_{J=0}, \tag{45}$$
where the indices '+' and '−' indicate that x_0 and y_0 are on the upper and on the lower branch of the contour.

- Keeping in mind that
$$\int_C d^4x\,\phi(x)J(x) = \int d^4x\,\phi_+(x)J_+(x) - \int d^4x\,\phi_-(x)J_-(x), \tag{46}$$
one finds
$$\frac{\delta}{\delta J_+(x_1)}\frac{\delta}{\delta J_-(x_2)}\exp\Big[i\int_C d^4x\,\phi(x)J(x)\Big] = i^2\,\phi_-(x_2)\,\phi_+(x_1)\,\exp\Big[i\int_C d^4x\,\phi(x)J(x)\Big]. \tag{47}$$

- Consequently,
$$\begin{aligned} i\Delta^>(x_1,x_2) \;&=\; Z^{-1}\int D\phi'(\mathbf{x})\,D\phi''(\mathbf{x})\,\rho\big[\phi'(\mathbf{x})\big|\phi''(\mathbf{x})\big] \\ &\times \int_{\substack{\phi(-\infty+i0^+,\mathbf{x})=\phi'(\mathbf{x})\\ \phi(-\infty-i0^+,\mathbf{x})=\phi''(\mathbf{x})}} \mathcal{D}\phi(x)\,\phi_-(x_2)\,\phi_+(x_1)\exp\Big[i\int_C d^4x\,\mathcal{L}_0(x)\Big]. \end{aligned} \tag{48}$$

Explicit form of free generating functional

- In case of vacuum generating functional, we have got the explicit form of free generating functional performing the Gaussian functional integration. Now, we cannot follow this path as the generating functional (42) depends on the boundary condition and further on the generating functional (42) should be integrated over the boundary fields $\phi'(\mathbf{x})$ and $\phi''(\mathbf{x})$.

- One observes that due to the field equation of motion
$$\big(\Box + m^2\big)\phi(x) - J(x) = 0, \tag{49}$$
the generating functional (40) obeys the following equation
$$\big(\Box + m^2\big)\frac{1}{i}\frac{\delta}{\delta J(x)}W_0[J] - J(x)\,W_0[J] = 0. \tag{50}$$

- Remembering the explicit form of the generating functional of vacuum QFT, one guesses the solution of the equation (50) as
$$\boxed{W_0[J] = \exp\Big[-\frac{i}{2}\int_C d^4x\int_C d^4y\,J(x)\Delta_0(x,y)J(y)\Big],} \tag{51}$$
where the normalization constant is absent because the condition $W_0[J=0]=1$ is trivially satisfied and $\Delta_0(x,y)$ is the free contour Green's function which satisfies the equations of motion
$$\big(\Box_x + m^2\big)\Delta_0(x,y) \;=\; -\delta_C^{(4)}(x,y), \tag{52}$$
$$\big(\Box_y + m^2\big)\Delta_0(x,y) \;=\; -\delta_C^{(4)}(x,y). \tag{53}$$

- A direct computation shows that the functional (51) solves the equation of motion (50).

Generating functional of self-interacting field

- Now the field self-interaction is taken into consideration by including the quartic term $-\frac{\lambda}{4!}\phi^4(x)$ in the Lagrangian density. The generating functional becomes

$$W[J] = \int \mathcal{D}\phi(x) \exp\Big\{ i\Big[S_0 + S_I + \int_C d^4x\, J(x)\phi(x) \Big]\Big\}, \tag{54}$$

where $\mathcal{N}$ is the normalization constant chosen in such a way that $W[J=0]=1$ and

$$S_I[\phi] = \int_C d^4x \mathcal{L}_I(x) = -\frac{\lambda}{4!}\int_C d^4x\, \phi^4(x). \tag{55}$$

- As in case of vacuum QFT, one uses of the observation that

$$\exp(\alpha x^2 + \beta x^4 + jx) = \exp\Big(\beta \frac{d^4}{dj^4}\Big) \exp(\alpha x^2 + jx), \tag{56}$$

and obtains the generating functional (54) as

$$\boxed{W[J] = \mathcal{N} \exp\Big\{ iS_I\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big\} W_0[J],} \tag{57}$$

where $W_0[J]$ is the generating functional of free fields (51), and

$$S_I\Big[\frac{1}{i}\frac{\delta}{\delta j}\Big] = -\frac{\lambda}{4!}\int_C d^4x \frac{\delta^4}{\delta j^4(x)}, \qquad \frac{1}{\mathcal{N}} = \exp\Big\{ iS_I\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big]\Big\} W_0[J]\Big|_{J=0}. \tag{58}$$

- The generating functional of the form (57) is used to obtain a perturbative expansion of any Green's function or, equivalently, to derive the Feynman rules.