Renormalization in Thermal QFT

In this Lecture, we return to the discussion of properties of the equilibrium boson gas that we conducted in Lectures III and IV, using the operator approach to the Matsubara formalism. Here we will apply the generating functional introduced in Lectures X and XI to derive higher order contributions to the free energy. We will deal with ultraviolet divergences, which require a renormalization. At the beginning, however, we discuss again the perturbative expansion of the partition function and we show how to get rid of disconnected diagrams.

The system under study is, as previously, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi(x)\partial_\mu \phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x). \tag{1}$$

Perturbative expansion of partition function

- As we remember, the path-integral representation of the partition function is

$$Z = \int \mathcal{D}\phi(x) \exp\left[S_0^E + S_I^E\right], \tag{2}$$

where

$$S_0^E[\phi] = \int_0^\beta d^4x\, \mathcal{L}_0^E(x) = -\frac{1}{2}\int_0^\beta d^4x\Big[\Big(\frac{\partial\phi(x)}{\partial\tau}\Big)^2 + \big(\nabla\phi(x)\big)^2 + m^2\phi^2(x)\Big], \tag{3}$$

$$S_I^E[\phi] = \int_0^\beta d^4x\, \mathcal{L}_I^E(x) = -\frac{\lambda}{4!}\int_0^\beta d^4x\, \phi^4(x), \tag{4}$$

with

$$\int_0^\beta d^4x \ldots \equiv \int_0^\beta d\tau \int d^3x \ldots \tag{5}$$

- Writing down the interaction action as

$$S_I^E = \lambda s_I, \tag{6}$$

where the coupling constant λ is singled out, the perturbative expansion of the partition function can be expressed in the following way

$$\begin{aligned} Z &= Z_0\big\langle e^{\lambda s_I}\big\rangle = Z_0\Big\langle 1 + \lambda s_I + \frac{\lambda^2}{2!}\, s_I^2 + \frac{\lambda^3}{3!}\, s_I^3 \ldots \Big\rangle \\ &= Z_0\Big(1 + \lambda\langle s_I\rangle + \frac{\lambda^2}{2!}\,\langle s_I^2\rangle + \frac{\lambda^3}{3!}\,\langle s_I^3\rangle \ldots\Big). \end{aligned} \tag{7}$$

where

$$\langle \ldots \rangle \equiv \frac{\int \mathcal{D}\phi(x)\, \ldots\, e^{S_0^E}}{\int \mathcal{D}\phi(x)\, e^{S_0^E}} = \frac{1}{Z_0}\int \mathcal{D}\phi(x)\, \ldots\, e^{S_0^E}. \tag{8}$$

- The first and the second order contributions to Z read

$$\langle s_I\rangle = \frac{1}{4!}\int_0^\beta d^4x\, \langle\phi^4(x)\rangle, \tag{9}$$

$$\langle s_I^2\rangle = \frac{1}{4!4!}\int_0^\beta d^4x \int_0^\beta d^4y\, \langle\phi^4(x)\,\phi^4(y)\rangle. \tag{10}$$

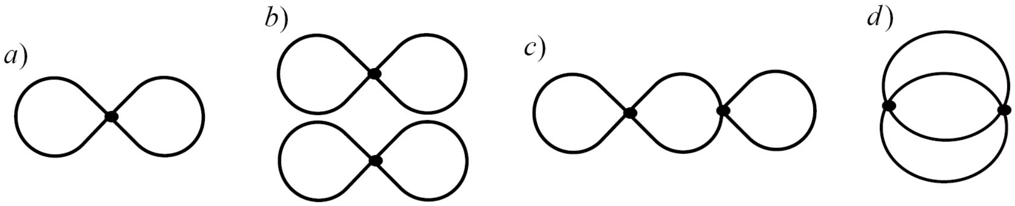

Figure 1: The diagrams representing the 1st and 2nd order contributions to the partition function (7)

- Using the free generating functional

$$W_0[J] = \exp\Big[-\frac{1}{2}\int_0^\beta d^4x_1 \int_0^\beta d^4x_2\, J(x_1)\Delta_0(x_1-x_2)J(x_2)\Big], \tag{11}$$

which is derived in Lecture XI, the expectation value $\langle\phi^4(x)\rangle$ is found as

$$\langle\phi^4(x)\rangle = \frac{\delta^4}{\delta J^4(x)} W_0[J]\Big|_{J=0} = 3\big(\Delta_0(x-x)\big)^2 = 3\langle\phi^2(x)\rangle^2 = 3\big(\Delta_0(0)\big)^2. \tag{12}$$

To compute the derivative one should realize that the only term from the expansion of the exponential function of generating functional (11) which contributes $\langle\phi^4(x)\rangle$ to is that one with four sources J. The terms with smaller number of Js vanish when the derivatives are computed. Those with bigger number of Js vanish when we set $J=0$.

- Similarly, one obtains

$$\begin{aligned}\langle\phi^4(x)\,\phi^4(y)\rangle &= \frac{\delta^4}{\delta J^4(x)}\frac{\delta^4}{\delta J^4(y)} W_0[J]\Big|_{J=0} \\ &= 9\big(\Delta_0(0)\big)^2 + 36\big(\Delta_0(0)\big)^2\big(\Delta_0(x-y)\big)^2 + 12\big(\Delta_0(x-y)\big)^4. \end{aligned}\tag{13}$$

- The diagrams representing the first and second order contributions to the partition function (7) are shown in Fig. 1. The diagram a) corresponds to $\langle\phi^4(x)\rangle$ given by the formula (12) while the diagrams b), c) and d) represent, respectively, the three terms in Eq. (13).
- We see that the diagram b) from Fig. 1 splits into two disconnected parts. This is so-called the disconnected diagram. Before we compute the first and second order contributions to the partition function, we are going to discuss how to eliminate the disconnected diagrams systematically.

Getting rid of disconnected diagrams

- Properties of a statistical system are encoded in its partition function. However, it appears more appropriate to compute from the beginning not the partition function Z but $\ln Z$ which is directly proportional to the free energy F as $F=-T\ln Z$.
- Keeping in mind that for $|x|\ll 1$ the logarithm is expanded as

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots, \tag{14}$$

the perturbative expansion of Z given by Eq. (7) gives the following expansion of $\ln Z$

$$\begin{aligned}\ln Z &= \ln Z_0 + \ln\Big(1 + \lambda\langle s_I\rangle + \frac{\lambda^2}{2!}\langle s_I^2\rangle + \frac{\lambda^3}{3!}\langle s_I^3\rangle \dots\Big) \\ &= \ln Z_0 + \lambda\langle s_I\rangle + \frac{\lambda^2}{2!}\big(\langle s_I^2\rangle - \langle s_I\rangle^2\big) + \frac{\lambda^3}{3!}\big(\langle s_I^3\rangle - 3\langle s_I\rangle\langle s_I^2\rangle + 2\langle s_I\rangle^3\big)\dots . \end{aligned}\tag{15}$$

- The expression $\big(\langle s_I^2\rangle - \langle s_I\rangle^2\big)$ corresponds to the sum of the diagrams b), c) and d) from Fig. 1 minus the square of the diagram a). Since the square of the diagram a) gives the diagram b), the expression $\big(\langle s_I^2\rangle - \langle s_I\rangle^2\big)$ corresponds to the sum of the diagrams c) and d). So, the disconnected diagram b) is eliminated.
- There is an analogous situation with the expression $\big(\langle s_I^3\rangle - 3\langle s_I\rangle\langle s_I^2\rangle + 2\langle s_I\rangle^3\big)$ which gives the sum of connected third order contributions to $\ln Z$.
- So, we conclude, the perturbative expansion of $\ln Z$ does not include the disconnected diagrams which contribute to Z.

First order contribution

- The first order correction to the partition function was already discussed in Lecture IV. We return to the issue performing calculation somewhat differently.
- The first order contribution corresponds to the diagram a) from Fig. 1 and it is given as

$$\ln Z\Big|_{(1)} = -\frac{\lambda}{4!}\,3\int_0^\beta d^4x\,\big(\Delta_0(x=0)\big)^2 = -\frac{\lambda}{8}\,\beta V\big(\Delta_0(x=0)\big)^2, \tag{16}$$

where the combinatorial factor 3 corresponds to a number of ways in which the four fields can be grouped in pairs.

- In Lecture IV we computed $\Delta_0(x=0)$, using the functions $\Delta_0^>(x),\ \Delta_0^<(x)$. In this way we avoided a summation over the Matsubara frequencies. Since the summation is unavoidable in computation of higher order contributions, it is instructive to compute the first order contribution in this way as well.
- Since the free temperature Green's function equals

$$\Delta_0(\tau,\mathbf{x}) = T\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,e^{-i(\omega_n\tau-\mathbf{p}\cdot\mathbf{x})}\Delta_0(\omega_n,\mathbf{p}), \tag{17}$$

where the Matsubara frequencies are

$$\omega_n \equiv 2\pi T\,n, \tag{18}$$

and

$$\Delta_0(\omega_n,\mathbf{p}) = \frac{1}{\omega_n^2+\omega_{\mathbf{p}}^2}, \tag{19}$$

the function of interest equals

$$\Delta_0(x=0) = T\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_n^2+\omega_{\mathbf{p}}^2}. \tag{20}$$

- Using the formula (see https://en.wikipedia.org/wiki/List_of_mathematical_series)

$$\sum_{n=0}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{2a}\,\mathrm{cth}(\pi a) + \frac{1}{2a^2}, \tag{21}$$

which can be rewritten as

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\,\mathrm{cth}(\pi a), \tag{22}$$

the sum over Matsubara frequencies in Eq. (20) is performed and the result is

$$\begin{aligned} \Delta_0(x=0) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\,\mathrm{cth}\Big(\frac{\omega_{\mathbf{p}}}{2T}\Big) &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\,\frac{e^{\beta\omega_{\mathbf{p}}}+1}{e^{\beta\omega_{\mathbf{p}}}-1} \\ &= \frac{1}{4\pi^2}\int_0^\infty \frac{dp\,p^2}{\sqrt{m^2+p^2}}\,\frac{1+e^{-\beta\sqrt{m^2+p^2}}}{1-e^{-\beta\sqrt{m^2+p^2}}}, \end{aligned} \tag{23}$$

where the trivial angular integral is taken in the last step.

- One checks that the integrand in Eq. (23) is finite in the infrared limit when $p \to 0$. This is obvious in case of $m > 0$. It is less obvious when $m = 0$. However, one checks that for $p \to 0$ the integrand is indeed finite. Specifically,

$$\lim_{p\to 0} p\,\frac{1+e^{-\beta p}}{1-e^{-\beta p}} = 2T. \tag{24}$$

So, we conclude that the integral (23) is <u>infrared safe</u>.

- The situation is different in the ultraviolet limit when $p \to \infty$. Since for $p \gg m$ and $p \gg T$ the integrand linearly grows with p, the integral in Eq. (23) is <u>quadratically divergent</u>.

- One observes that the ultraviolet divergence is present in the zero temperature limit of $\Delta_0(x=0)$ when $\beta \to \infty$. So, we split the integral into the vacuum and medium parts as

$$\Delta_0(x=0) = \frac{1}{4\pi^2}\int_0^\infty \frac{dp\,p^2}{\sqrt{m^2+p^2}}\,\frac{1+e^{-\beta\sqrt{m^2+p^2}}}{1-e^{-\beta\sqrt{m^2+p^2}}} = \Delta_0^{\rm vac}(x=0) + \Delta_0^{\rm med}(x=0), \tag{25}$$

where

$$\Delta_0^{\rm vac}(x=0) \equiv \frac{1}{4\pi^2}\int_0^\infty \frac{dp\,p^2}{\sqrt{m^2+p^2}}, \tag{26}$$

$$\Delta_0^{\rm med}(x=0) \equiv \frac{1}{2\pi^2}\int_0^\infty \frac{dp\,p^2}{\sqrt{m^2+p^2}}\,\frac{1}{e^{\beta\sqrt{m^2+p^2}}-1}. \tag{27}$$

We see that the medium part is regular in the ultraviolet limit while the vacuum contribution quadratically diverges. So, this is the ultraviolet divergence of vacuum QFT.

- When we first encountered the divergence in Lecture IV, we simply eliminated $\Delta^{\rm vac}(x=0)$, arguing that the vacuum contribution should not influence thermodynamic characteristics of the system. However, the procedure is not fully satisfactory, as the divergent integral produces a temperature dependent contribution to the partition function. Indeed, the first order contribution to the partition function (16) equals

$$\ln Z\Big|_{(1)} = -\frac{\lambda}{8}\,\beta V\Big(\Delta_0^{\rm vac}(x=0) + \Delta_0^{\rm med}(x=0)\Big)^2. \tag{28}$$

- We see that the divergent function $\Delta_0^{\rm vac}(x=0)$ influences the system's thermodynamics. So, the problem of ultraviolet divergences requires a more careful analysis.

Renormalization

- This is generally true with statistical QFT that ultraviolet divergences, in contrast to infrared ones, are caused by the vacuum sector of the theory. The physical reason is the following. The ultraviolet divergences occur due to momenta p which are much bigger than the temperature T or equivalently, at distances x much smaller that T^{-1}. In a relativistic gas, where T is much bigger than masses of gas constituents, the particle density ρ is of order T^3. (It is sufficient here to refer to the dimensional argument.) Consequently, the inter-particle spacing is of order $\rho^{-1/3} \sim T^{-1}$. So, ultraviolet divergences occur at distances x much smaller than the inter-particle spacing. Therefore, a medium plays no role here and a renormalization procedure to tame the divergences is exactly as in vacuum QFT. So, we discuss the procedure only briefly.

Vacuum sector of thermal field theory

- It is important to realize that the zero-temperature sector of thermal field theory or, equivalently, the zero-temperature limit of the theory, which is plagued with ultraviolet divergences, corresponds to the Euclidean formulation of vacuum QFT that is the Minkowski space is replaced by the Euclidean one with the diagonal metric tensor $(-,-,-,-)$ not $(+,-,-,-)$.
- Since the spacing of Matsubara frequencies goes to zero at vanishing temperature, the zeroth component of four-momentum, which is discrete and represented by Matsubara frequencies at a finite temperature, becomes a continuous variable usually denoted as p_4 at $T=0$ and it equals $p_4 = ip_0$ where p_0 is the zeroth component of four-momentum in the Minkowski space. In the zero-temperature limit the sum over Matsubara frequencies is replaced by the integral over p_4 as

$$T \sum_{n=-\infty}^{\infty} \ldots \longrightarrow \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \ldots \tag{29}$$

- The zero-temperature or vacuum free Green's function in momentum space is

$$\Delta_0^{\text{vac}}(p) = \frac{1}{p_4^2 + \mathbf{p}^4 + m^4}, \tag{30}$$

  where $p = (p_4, \mathbf{p})$.
- At $T = 0$ the integral over τ from 0 to β is replaced by the integral from $-\infty$ to ∞. Since the integrand is periodic in τ with the period β, the integral from 0 to β is first replaced by the integral from $-\beta/2$ to $\beta/2$ and then the limit $\beta \to \infty$ is taken. So, at the vanishing temperature the following replacement is applied

$$\int_0^\beta d^4x \ldots \equiv \int_0^\beta d\tau \int d^3x \ldots \longrightarrow \int d^4x \ldots \equiv \int_{-\infty}^{\infty} d\tau \int d^3x \ldots \tag{31}$$

The idea of renormalization

- The procedure of renormalization consists in absorbing infinite expressions into the physical or measurable parameters of a given theory. In case of the scalar self-interacting field these are the coupling constant and mass. However, one introduces an important distinction. The parameters λ and m, which enter the Lagrangian density, are interpreted as bare parameters, and further on these parameters are denoted as λ_B and m_B. The physical or renormalized parameters are those into which divergent integrals are absorbed. The procedure is performed order by order in perturbative expansion and the bare parameters are expressed by the renormalized ones as

$$m_B^2 = m^2 + \underbrace{\lambda c_{(1)} + \lambda^2 c_{(2)} + \dots}_{\equiv \delta m^2}, \tag{32}$$

$$\lambda_B = \lambda + \underbrace{\lambda^2 d_{(2)} + \lambda^3 d_{(3)} + \dots}_{\equiv \delta\lambda}, \tag{33}$$

  where the coefficients $c_{(n)},\ d_{(n)}$ include the divergent integrals.

- We note that a renormalization of field amplitudes is required at higher orders of perturbative expansion. However, we not discuss it.

- The renormalized mass and coupling constant are found as $m^2 = m_B^2 - \delta m^2$ and $\lambda = \lambda_B - \delta\lambda$. So, the physical parameters m and λ are expressed through the quantities m_B^2, δm^2 and λ_B, $\delta\lambda$ which are formally infinite. However, we can live with it, because these quantities are of purely theoretical significance.

- In order to perform the renormalization procedure, a renormalization scheme should be chosen. The scheme determines a method to define physical quantities, in our case the mass and coupling constant. In our further considerations, we will use the most traditional scheme in which the mass and coupling constant are defined at a vanishing momentum. However, it should be stressed that other choices are often applied.

- Finally, we note that the program of renormalization can be successfully realized only for a class of theories called renormalizable. The class includes the theory of self-interacting scalar field that we consider here. In renormalizable theories, a finite number of physical quantities is sufficient to absorb divergences in arbitrary high order of the perturbation expansion.

- In the first order of perturbative expansion the divergences should be fully absorbed in the renormalized mass as $\lambda_B = \lambda$. So, we have to define the particle's mass.

- In vacuum QFT the particle's mass is defined as an energy corresponding to a pole of particle's propagator at vanishing momentum. In the imaginary time formalism the free propagator (30) has no pole at real energies. So, the definition of mass should be modified. However, we will not formalize the definition, but instead, we will remember that going from the Euclidean to Minkowski space-time the mass corresponds to a pole of the particle's propagator at vanishing momentum.

- To systematically perform the procedure of subtraction of divergent integrals one rewrites the Lagrangian density (1) as

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) - \frac{1}{2}\delta m^2\phi^2(x) - \frac{\delta\lambda}{4!}\phi^4(x), \tag{34}$$

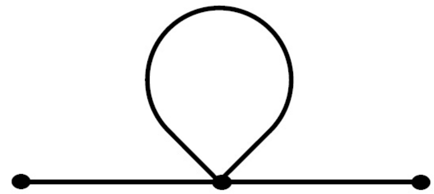

Figure 2: The first order correction to the Green's function

where the two counterterms are included. It should be stressed that λ and m are now physical parameters.

The first order mass counterterm

- We are going to derive here the first order mass counterterm δm^2. As we already noted, $\delta\lambda = 0$ at the first order.
- Since the physical mass is inferred from the propagator, we compute it at the first order of perturbative expansion. The first order contribution, which is represented by the diagram shown in Fig. 2, equals

$$\begin{aligned} \Delta^{\rm vac}_{(1)}(x-y) &= -\frac{\lambda}{4!}\,12\int d^4z\,\Delta^{\rm vac}_0(x-z)\,\Delta^{\rm vac}_0(z-z)\,\Delta^{\rm vac}_0(z-y) \\ &= -\frac{\lambda}{2}\,\Delta^{\rm vac}_0(z=0)\int d^4z\,\Delta^{\rm vac}_0(x-z)\,\Delta^{\rm vac}_0(z-y), \end{aligned} \tag{35}$$

where the combinatorial factor of 12 reflects the fact that there are 4 ways to pair the field $\phi(x)$ with the four fields $\phi(z)\,\phi(z)\,\phi(z)\,\phi(z)$ and 3 ways to pair the $\phi(y)$ with the remaining three fields $\phi(z)\,\phi(z)\,\phi(z)$. So, the combinatorial factor is $4\cdot 3 = 12$.

- Going to the momentum space, Eq. (35) becomes

$$\Delta^{\rm vac}_{(1)}(p) = -\frac{\lambda}{2}\,\Delta^{\rm vac}_0(z=0)\big(\Delta^{\rm vac}_0(p)\big)^2, \tag{36}$$

- Treating the mass counterterm in the Lagrangian density (34) as the interaction term, one finds the additional first order contribution

$$\delta\Delta^{\rm vac}_{(1)}(x-y) = -\frac{1}{2}\,2\,\delta m^2\int d^4z\,\Delta^{\rm vac}_0(x-z)\,\Delta^{\rm vac}_0(z-y), \tag{37}$$

which in the momentum space reads

$$\delta\Delta^{\rm vac}_{(1)}(p) = -\delta m^2\big(\Delta^{\rm vac}_0(p)\big)^2. \tag{38}$$

- The complete first order correction to the Green's function is

$$\Delta^{\rm vac}_{(1)}(p) = -\Big(\frac{\lambda}{2}\,\Delta^{\rm vac}(z=0) + \delta m^2\Big)\big(\Delta^{\rm vac}_0(p)\big)^2. \tag{39}$$

- Introducing the self energy $\Pi^{\rm vac}$, which, as we remember, is defined through the Dyson-Schwinger equation as

$$\Delta^{\rm vac}(p) = \Delta^{\rm vac}_0(p) - \Delta^{\rm vac}_0(p)\Pi^{\rm vac}(p)\,\Delta^{\rm vac}(p), \tag{40}$$

Eq. (39) is rewritten as

$$\Delta^{\text{vac}}_{(1)}(p) = -\Pi^{\text{vac}}_{(1)}\big(\Delta^{\text{vac}}_0(p)\big)^2, \tag{41}$$

where

$$\Pi^{\text{vac}}_{(1)} = \frac{\lambda}{2}\,\Delta^{\text{vac}}(z=0) + \delta m^2. \tag{42}$$

As we see, the self energy (42) is independent of momentum.

- Resumming the first order contribution, as we already did several times, one gets the resummed propagator

$$\Delta^{\text{vac}}(p) = \frac{1}{p_4^2 + \mathbf{p}^2 + m^2 + \Pi^{\text{vac}}_{(1)}}. \tag{43}$$

- Since m is the physical mass, δm^2 should be chosen in such a way that $\Pi^{\text{vac}}_{(1)} = 0$. This means

$$\delta m^2 = -\frac{\lambda}{2}\,\Delta^{\text{vac}}_0(z=0). \tag{44}$$

- Let us compute $\Delta^{\text{vac}}_0(z=0)$, which is

$$\Delta^{\text{vac}}_0(z=0) = \int \frac{d^4 p_E}{(2\pi)^4}\,\frac{1}{p_E^2 + m^2}, \tag{45}$$

where $p_E \equiv (p_4, \mathbf{p})$ and $p_E^2 = p_4^2 + \mathbf{p}^2$.

- The integral in Eq. (45) is most easily computed in four-dimensional spherical coordinates as

$$\Delta^{\text{vac}}_0(z=0) = \int \frac{d^3\Omega}{(2\pi)^4}\int_0^\Lambda \frac{dp\,p^3}{p^2+m^2}, \tag{46}$$

where $p \equiv \sqrt{p_E^2} = \sqrt{p_4^2 + \mathbf{p}^2}$ and the upper momentum cut-off Λ offers the most straightforward method to regularize the integral.

- The angular integral is found as

$$\int d^{n-1}\Omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{47}$$

which equals $2\pi^2$ for $n = 4$.

<u>Exercise</u>: Derive the integral formula (47).

- Since $\Lambda \gg m$, we can ignore m in the integrand (46) and we get

$$\int_0^\Lambda \frac{dp\,p^3}{p^2+m^2} = \frac{1}{2}\,\Lambda^2. \tag{48}$$

- Substituting the results (47, 48) in Eq. (46), one finds

$$\Delta^{\text{vac}}_0(z=0) = \frac{\Lambda^2}{16\pi^2}, \tag{49}$$

and using Eq. (44), we finally obtain

$$\delta m^2 = -\frac{\lambda\Lambda^2}{32\pi^2}. \tag{50}$$

Digression
Let us compute $\Delta_0^{\rm vac}(z=0)$ starting with the formula (26). Cutting off the integral with Λ' and assuming that $\Lambda' \gg m$, one finds

$$\Delta_0^{\rm vac}(x=0) = \frac{1}{4\pi^2}\int_0^{\Lambda'} \frac{dp\,p^2}{\sqrt{m^2+p^2}} = \frac{1}{4\pi^2}\int_0^{\Lambda'} dp\,p = \frac{\Lambda'^2}{8\pi^2}, \tag{51}$$

which differs from the result (49) for $\Lambda' = \Lambda$. At first glance the difference is surprising but it can be easily understood: the integrals (26) and (45) are simply different. However, one can get the integral (26) performing first the integration over p_4 in Eq. (45). Indeed, one finds

$$\begin{aligned}\Delta_0^{\rm vac}(z=0) &= \int \frac{d^3p}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{dp_4}{p_4^2+\mathbf{p}^2+m^2} = \int \frac{d^3p}{(2\pi)^4}\frac{1}{\sqrt{m^2+\mathbf{p}^2}}\,{\rm arctg}\frac{p_4}{\sqrt{m^2+\mathbf{p}^2}}\Big|_{-\infty}^{\infty} \\ &= \pi\int\frac{d^3p}{(2\pi)^4}\frac{1}{\sqrt{m^2+\mathbf{p}^2}} = \frac{1}{4\pi^2}\int_0^{\infty}\frac{dp\,p^2}{\sqrt{m^2+p^2}}.\end{aligned} \tag{52}$$

The renormalized first order contribution to $\ln Z$

- Now, we are going to show that the mass counterterm in the Lagrangian density (34) with δm^2 given by Eq. (50), will make finite the first order contribution to $\ln Z$ given by Eq. (16) or (28).
- Due to the mass counterterm in the Lagrangian, we have an additional first order contribution to $\ln Z$

$$\begin{aligned}\delta \ln Z\Big|_{(1)} &= -\frac{1}{2}\,\delta m^2\int_0^{\beta} d^4x\,\Delta_0(x=0) = -\frac{1}{2}\,\delta m^2\,\beta V\,\Delta_0(x=0), \\ &= -\frac{1}{2}\,\delta m^2\,\beta V\Big(\Delta_0^{\rm vac}(x=0)+\Delta_0^{\rm med}(x=0)\Big).\end{aligned} \tag{53}$$

- Consequently, the complete first order contribution to $\ln Z$ equals

$$\begin{aligned}\ln Z\Big|_{(1)} &= -\frac{\lambda}{8}\,\beta V\Big(\Delta_0^{\rm vac}(x=0)+\Delta_0^{\rm med}(x=0)\Big)^2 \\ &\quad -\frac{1}{2}\,\delta m^2\,\beta V\Big(\Delta_0^{\rm vac}(x=0)+\Delta_0^{\rm med}(x=0)\Big),\end{aligned} \tag{54}$$

which, using Eq. (44), can be rewritten as

$$\begin{aligned}\frac{1}{\lambda\beta V}\,\ln Z\Big|_{(1)} &= -\frac{1}{8}\Big(\Delta_0^{\rm vac}(x=0)+\Delta_0^{\rm med}(x=0)\Big)^2 \\ &\quad +\frac{1}{4}\,\Delta_0^{\rm vac}(x=0)\Big(\Delta_0^{\rm vac}(x=0)+\Delta_0^{\rm med}(x=0)\Big) \\ &= -\frac{1}{8}\Big(\Delta_0^{\rm med}(x=0)\Big)^2+\frac{1}{8}\Big(\Delta_0^{\rm vac}(x=0)\Big)^2.\end{aligned} \tag{55}$$

One observes that the mass counterterm cancels the disturbing contribution to $\ln Z$ which is proportional to $\Delta_0^{\rm vac}(x=0)\,\Delta_0^{\rm med}(x=0)$.

- Assuming that $T \gg m$, one computes $\Delta_0^{\rm med}(x=0)$ given by Eq. (27) as

$$\Delta_0^{\rm med}(x=0) = \frac{1}{2\pi^2}\int_0^{\infty}\frac{dp\,p}{e^{\beta p}-1} = \frac{T}{2\pi^2}\underbrace{\int_0^{\infty}\frac{dx\,x}{e^x-1}}_{=\frac{\pi^2}{6}} = \frac{T^2}{12}. \tag{56}$$

- Substituting the results (49, 56) into Eq. (55), one finds

$$\frac{1}{\lambda\beta V}\,\ln Z\Big|_{(1)} = -\frac{T^4}{1152} + \frac{\Lambda^4}{2048\pi^4}, \tag{57}$$

where the divergent term is still present.

- To understand a role of the divergent term, let us consider the partition function. Keeping in mind that $\ln Z_0 = (\pi^2 V\,T^3)/90$, we have

$$\ln Z = \ln Z_0 + \ln Z\Big|_{(1)} = \frac{\pi^2 V\,T^3}{90}\Big[1 - \frac{5\lambda}{64\pi^2}\Big] + \frac{\lambda V \Lambda^4}{2048\pi^4 T}. \tag{58}$$

- With the partition function (58), the system's energy U, free energy $F = U - TS$ and pressure p, which are defined as

$$U \equiv -\frac{d}{d\beta}\ln Z(T), \qquad F \equiv -T\ln Z(T), \qquad p = -\Big(\frac{\partial F}{\partial V}\Big)_T, \tag{59}$$

are

$$U = \frac{\pi^2 V\,T^4}{30}\Big[1 - \frac{5\lambda}{64\pi^2}\Big] - \frac{\lambda V \Lambda^4}{2048\pi^4}, \tag{60}$$

$$F = -\frac{\pi^2 V\,T^4}{90}\Big[1 - \frac{5\lambda}{64\pi^2}\Big] - \frac{\lambda V \Lambda^4}{2048\pi^4}, \tag{61}$$

$$p = \frac{\pi^2 T^4}{90}\Big[1 - \frac{5\lambda}{64\pi^2}\Big]. \tag{62}$$

- One observes that the divergent term does not enter the pressure but it shifts the energy and free energy by a constant value which corresponds to the vacuum energy density. As long as gravity is not considered, only energy differences are measurable and thus the divergent term can be safely ignored. If we included gravity, the vacuum energy, we encountered, would be absorbed in the renormalized cosmological constant which is interpreted as the vacuum energy density.

- We also note without the cancellation of the divergent terms in Eq. (55), we would have a contribution to $\ln Z$ which is proportional to $\lambda V T \Lambda^2$. Such a term provides the temperature dependent contributions to the system's energy, free energy and pressure.

- We conclude the lecture by saying that the renormalization of thermal field theory is performed as in the vacuum counterpart theory and the renormalization does not influence thermodynamic characteristics obtained by means of perturbative expansion. To get rid of ultraviolet contribution it is usually sufficient to simply subtract vacuum contributions. The theory parameters – masses and coupling coupling constants – are then treated as renormalized. We will follow this pragmatic strategy in the subsequent lectures.