Resummation of infrared divergent diagrams

In this Lecture we continue the discussion on the perturbative expansion of the partition function started in Lecture XII. We analyze the second order contributions. It appears that one contribution is badly infrared divergent. We show that a resummation of a class of diagrams removes the divergence. We also discuss a reorganized perturbative expansion where the resummation is achieved by using a resummed Green's function.

Second order contributions to $\ln Z$

The second order contributions to $\ln Z$ are represented by the two diagrams shown in Fig. 1. We first discuss the diagram a) and then the diagram b).

Diagram a)

- The contribution corresponding to the diagram a) from Fig. 1 is

$$\ln Z\Big|_{(2a)} = \lambda^2 \underbrace{\frac{1}{(4!)^2}\frac{72}{2!}}_{=\frac{1}{16}} \int_0^\beta d^4x \int_0^\beta d^4y\, \Delta(x=0)\, \Delta(x-y)\, \Delta(y-x)\, \Delta(y=0), \qquad (1)$$

  where $x_0 = \tau_x,\ y_0 = \tau_y$ and the combinatorial factor of 72 is found as follows. There are 3! ways to choose a pair of fields from the four fields $\phi(x)\,\phi(x)\,\phi(x)\,\phi(x)$ or $\phi(y)\,\phi(y)\,\phi(y)\,\phi(y)$. There are two ways to pair two unpaired fields $\phi(x)\,\phi(x)$ with two unpaired field $\phi(y)\,\phi(y)$. So, the combinatorial factor is $(3!)^2 2 = 72$.

- Eq. (1) is rewritten as

$$\ln Z\Big|_{(2a)} = \frac{\lambda^2}{16}\big(\Delta(x=0)\big)^2 I, \qquad (2)$$

  where

$$I \equiv \int_0^\beta d^4x \int_0^\beta d^4y\, \Delta(x-y)\, \Delta(y-x). \qquad (3)$$

  As we remember, $\Delta(x=0) = T^2/12$. So, the integral I is the quantity to be computed.

- Expressing the function $\Delta(x)$ as

$$\Delta(\tau, \mathbf{x}) = T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}\, e^{-i(\omega_n \tau_x - \mathbf{p}\cdot\mathbf{x})} \Delta(\omega_n, \mathbf{p}), \qquad (4)$$

  where

$$\Delta_0(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \omega_{\mathbf{p}}^2} \qquad (5)$$

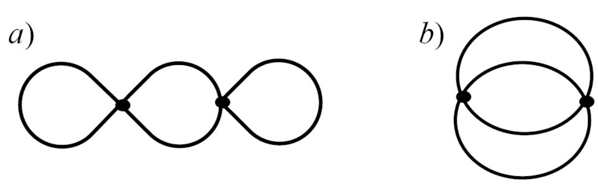

Figure 1: The diagrams representing the 2nd order contributions to $\ln Z$

with $\omega_n \equiv 2\pi T\, n$, the quantity (3) becomes

$$I = T^2 V \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \int_0^\beta d\tau_x \int_0^\beta d\tau_y\, e^{-i(\omega_n - \omega_{n'})(\tau_x - \tau_y)} \Delta(\omega_n, \mathbf{p})\, \Delta(\omega_{n'}, \mathbf{p}). \tag{6}$$

- Since

$$\int_0^\beta d\tau\, e^{-i(\omega_n - \omega_n')\tau} = \beta\, \delta^{n\,n'}, \tag{7}$$

we get

$$I = V \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \big(\Delta(\omega_n, \mathbf{p})\big)^2. \tag{8}$$

- Using the function (5), Eq. (8) gives

$$I = V \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\omega_n^2 + \omega_{\mathbf{p}}^2)^2} = \frac{V}{2\pi^2} \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{dp\, p^2}{(\omega_n^2 + m^2 + p^2)^2}$$

$$= \frac{V}{2^5 \pi^6 T^4} \int_0^\infty \sum_{n=-\infty}^{\infty} \frac{dp\, p^2}{(n^2 + a^2)^2}, \tag{9}$$

where the trivial angular integral is taken and $a \equiv \frac{\sqrt{p^2+m^2}}{2\pi T}$.

- The formula, which is required to perform the sum in Eq. (9), can be obtained from the previously used formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \operatorname{cth}(\pi a), \tag{10}$$

observing that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2} = -\frac{d}{da^2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\frac{d}{da^2} \frac{\pi}{a} \operatorname{cth}(\pi a) = -\frac{\pi}{2a} \frac{d}{da} \frac{\operatorname{cth}(\pi a)}{a}, \tag{11}$$

which gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + a^2)^2} = \frac{\pi}{2a^3} \operatorname{cth}(\pi a) + \frac{\pi^2}{2a^2} \frac{1}{\sinh^2(\pi a)}. \tag{12}$$

- Using the formula (12), one finds

$$I = \frac{V}{2^3 \pi^2 T^2} \int_0^\infty \frac{dp\, p^2}{m^2 + p^2} \Bigg[\frac{T}{\sqrt{p^2+m^2}} \frac{e^{\frac{\beta}{2}\sqrt{p^2+m^2}} + e^{-\frac{\beta}{2}\sqrt{p^2+m^2}}}{e^{\frac{\beta}{2}\sqrt{p^2+m^2}} - e^{-\frac{\beta}{2}\sqrt{p^2+m^2}}}$$

$$+ \frac{2}{\big(e^{\frac{\beta}{2}\sqrt{p^2+m^2}} - e^{-\frac{\beta}{2}\sqrt{p^2+m^2}}\big)^2} \Bigg]. \tag{13}$$

- Let us consider, how the integral (13) behaves in the infrared domain when $p \to 0$. If m is nonzero and $p \ll m$, the integrand (13) is approximated as

$$\frac{p^2}{m^2} \left[\frac{T}{m} \frac{e^{\frac{\beta m}{2}} + e^{-\frac{\beta m}{2}}}{e^{\frac{\beta m}{2}} - e^{-\frac{\beta m}{2}}} + \frac{2}{\big(e^{\frac{\beta m}{2}} - e^{-\frac{\beta m}{2}}\big)^2} \right] \tag{14}$$

and it vanishes as $p \to 0$. So, the integral is infrared save for $m \neq 0$.

- If $m = 0$, the integrand is approximated as $4T^2/p^2$ when $p \to 0$. So, the integral is infrared divergent.
- As we will see, one has to resum a whole class of infrared divergent diagrams to tame the divergence. As a result, one obtains a contribution which, however, is not of the order λ^2 but $\lambda^{3/2}$.

Diagram b)

- The contribution corresponding to the diagram b) from Fig. 1 is

$$\ln Z\Big|_{(2b)} = \lambda^2 \underbrace{\frac{1}{(4!)^2}\frac{4!}{2!}}_{=\frac{1}{48}} \int_0^\beta d^4x \int_0^\beta d^4y\, \Delta(x-y)\,\Delta(x-y)\,\Delta(y-x)\,\Delta(y-x), \tag{15}$$

where the combinatorial factor of 4! equals the number of ways to pair each of four fields $\phi(x)\,\phi(x)\,\phi(x)\,\phi(x)$ with four fields $\phi(y)\,\phi(y)\,\phi(y)\,\phi(y)$.

- A computation of the diagram b) appears truly difficult. However, we are mostly interested in one specific question whether the diagram provides a infrared divergent contribution similarly to the diagram a).
- To study the infrared domain of the contribution (15), it is sufficient to know how the Green's function $\Delta(\tau, \mathbf{r})$ behaves at large distances. The variable τ is limited as $0 \le \tau \le \beta$ and it is never big. So, we consider $\Delta(\tau, \mathbf{r})$ at large $\mathbf{r}$. One realizes that the limit is given by the formula (4) with the zero Matsubara frequency. Thus, Eq. (4) gives

$$\Delta^{n=0}(\mathbf{r}) = T \int \frac{d^3p}{(2\pi)^3}\, \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 + m^2} = \frac{T}{4\pi}\, \frac{e^{-mr}}{r}, \tag{16}$$

where $r \equiv |\mathbf{r}|$. We stress that the function (16) is independent of τ.

<u>Exercise</u>: Derive the formula (16).

- Substituting the Green's function (16) into Eq. (15) and keeping in mind that $\Delta(x-y) = \Delta(y-x)$ in case of a real field, one finds the infrared domain of the contribution (15) as

$$\begin{aligned} \ln Z\Big|_{(2b)}^{\text{infrared}} &= \frac{\lambda^2}{48}\beta^2 \int d^3x \int d^3y\, \big(\Delta^{n=0}(\mathbf{x}-\mathbf{y})\big)^4 = \frac{\lambda^2}{48}\beta^2 \int d^3R \int d^3r\, \big(\Delta^{n=0}(\mathbf{r})\big)^4 \\ &= \frac{\lambda^2}{48} V\beta^2 \int d^3r\, \big(\Delta^{n=0}(\mathbf{r})\big)^4 = \frac{\lambda^2}{48}\frac{VT^2}{(4\pi)^3} \int_{r_0}^\infty \frac{dr\, e^{-4mr}}{r^2}, \end{aligned} \tag{17}$$

where $r_0 \gg m^{-1}$ and $r_0 \gg T^{-1}$. One sees that the integral (17) is convergent in the long distance limit not only form $m > 0$ but for $m = 0$ as well.

- Since the contribution (15) is infrared finite it is of order λ^2.

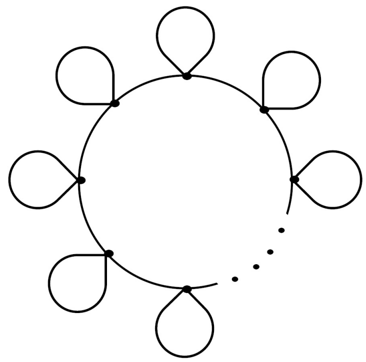

Figure 2: The ring diagram

Resummation

We have found a diagram which is infrared divergent in case of massless fields. The situation is actually typical for gauge theories like QED or QCD where we deal with massless gauge bosons. The resummation program, which offers a method to tame the infrared divergences, can be realized in two ways which look rather different at a first glance. One either resums a class of infrared divergent diagrams computed in a massless theroy and gets a finite contribution or one rearranges the perturbative expansion in such a way that a dynamically generated mass is included into a free propagator. Below we discuss both approaches one by one.

Since the infrared divergences of interest occur for massless field we set $m = 0$ in this section.

Resummation of ring diagrams

- One realizes that the infrared divergence analogous but even more severe to that of the diagram a) from Fig. 1 occurs for a whole class of diagrams which are shown in Fig. 2.
- The ring diagram of order λ^N with $N > 2$ gives the following contribution to $\ln Z$

$$\ln Z\Big|_{(N\,\mathrm{ring})} = \Big(-\frac{\lambda}{4!}\Big)^N \frac{C_N}{N!} \int_0^\beta d^4x_1 \int_0^\beta d^4x_2 \cdots \int_0^\beta d^4x_N \times \Delta(0)\,\Delta(x_1 - x_2)\,\Delta(0)\Delta(x_2 - x_3)\,\Delta(0)\ldots\Delta(x_N - x_1), \tag{18}$$

  where C_N is the combinatoric factor computed as follows. There are 3! ways to choose a pair of fields out of four fields in each vertex. A factor 2 gives number of possible pairings of the unpaired fields in two neighboring vertices. There are $\frac{1}{2}(N-1)!$ ways to order the vertices along the ring. Thus, one finds $C_N = \frac{1}{2}(N-1)!(2\cdot 3!)^N$.

- Going to the momentum space, the formula (18) becomes

$$\ln Z\Big|_{(N\,\mathrm{ring})} = \frac{1}{2}\Big(-\frac{\lambda}{4!}\Big)^N \frac{(2\cdot 3!)^N}{N} \big(\Delta(x=0)\big)^N T^N \beta^N V \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \big(\Delta(\omega_n, \mathbf{p})\big)^N, \tag{19}$$

  where $N-1$ four-dimensional integrals, which are trivial due to the delta functions, are taken.

- Since $\Delta(x=0) = T^2/12$ (with appropriately renormalized $\Delta(x)$), Eq. (20) is rewritten as

$$\ln Z\Big|_{(N\,\mathrm{ring})} = \frac{1}{2} V \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{\big(\Pi\,\Delta(\omega_n, \mathbf{p})\big)^N}{N}, \tag{20}$$

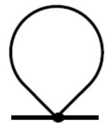

Figure 3: The first order contribution to the self energy

where

$$\Pi \equiv \lambda \frac{T^2}{24}. \tag{21}$$

- Actually, Π is the first order self energy corresponding to the diagram shown in Fig. 3. The self energy is renormalized that is the vacuum contribution is subtracted.
- Keeping in mind the Taylor expansion of the logarithm, which is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{N=1}^{\infty} \frac{(-x)^N}{N}, \tag{22}$$

the ring contributions (20) can be summed from $N=2$ to $N=\infty$ and the result is

$$\ln Z\Big|_{\Sigma\ \mathrm{ring}} = \frac{1}{2} V \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \Big[\ln\Big(1 + \Pi\, \Delta(\omega_n, \mathbf{p})\Big) - \Pi\, \Delta(\omega_n, \mathbf{p})\Big]. \tag{23}$$

- Using the explicit form of the temperature Green's function (5) and taking the trivial angular integral, the formula (23) becomes

$$\ln Z\Big|_{\Sigma\ \mathrm{ring}} = \frac{V}{4\pi^2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dp\, p^2 \Big[\ln\Big(1 + \frac{\Pi}{(2\pi T n)^2 + p^2}\Big) - \frac{\Pi}{(2\pi T n)^2 + p^2}\Big]. \tag{24}$$

- We observe that the integral (24) is regular in both infrared and ultraviolet limits for any n including $n=0$. In the infrared limit the two terms in Eq. (24) are regular independently from each other but in the ultraviolet limit the two terms partially cancel each other to decay as p^{-2}.
- We simplify the integral (24) performing the partial integration as

$$\begin{aligned} \ln Z\Big|_{\Sigma\ \mathrm{ring}} &= \frac{V}{12\pi^2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dp\, \frac{dp^3}{dp} \Big[\ln\Big(1 + \frac{\Pi}{(2\pi T n)^2 + p^2}\Big) - \frac{\Pi}{(2\pi T n)^2 + p^2}\Big] \\ &= \frac{V}{6\pi^2} \Pi^2 \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dp\, p^4}{\big((2\pi T n)^2 + p^2\big)^2 \big((2\pi T n)^2 + p^2 + \Pi\big)}. \end{aligned} \tag{25}$$

- Since $(2\pi T n)^2 \gg \Pi$ for $n \neq 0$, Π can be ignored in the integrand (25) in such a case. Therefore, the sum of ring diagrams with the term $n=0$ excluded is of order λ^2 as Π is of order λ.
- The situation is different when $n=0$, which corresponds to the static zero mode, because Π cannot be ignored in the integrand (25). Computing the integral (25) for $n=0$, one finds

$$\ln Z\Big|_{\Sigma\ \mathrm{ring}}^{n=0} = \frac{V}{6\pi^2} \Pi^2 \int_0^{\infty} \frac{dp}{p^2 + \Pi} = \frac{V}{6\pi^2} \Pi^{3/2} \underbrace{\int_0^{\infty} \frac{dx}{1+x^2}}_{=\frac{\pi}{2}} = \frac{V}{12\pi} \Pi^{3/2}. \tag{26}$$

- Using the formula (21), one finally obtains

$$\ln Z\Big|_{\Sigma\ \mathrm{ring}}^{n=0} = \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{12\pi}. \tag{27}$$

- The zero mode provides the contribution of the order $\lambda^{3/2}$ in contrast to the nonzero modes which contribute at the order λ^2.

- An appearance of the contribution with fractional power of λ shows that $\ln Z$ is a non-analytical function of λ.

- Collecting the first three terms of the prerturbative expansion of $\ln Z$, we have

$$\begin{aligned}\ln Z &= \ln Z_0 + \ln Z\Big|_{(1)} + \ln Z\Big|_{\Sigma\ \mathrm{ring}}^{n=0} + \dots \\ &= \frac{\pi^2 V T^3}{90} - \lambda\frac{VT^3}{1152} + \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{12\pi} + \dots \\ &= \frac{\pi^2 V T^3}{90}\Big[1 - \frac{15}{8}\Big(\frac{\lambda}{4!\pi^2}\Big) + \frac{15}{2}\Big(\frac{\lambda}{4!\pi^2}\Big)^{3/2} + \dots\Big]. \end{aligned}\tag{28}$$

Resummation through the effective mass

- The second method of resumation relies on using in the preturbative expansion not the free Green's function but the function where the lowest order correction is resummed to infinite order. As we remember, the function is of the form

$$\Delta(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + \Pi_{(1)}(p)}. \tag{29}$$

- Since the self energy $\Pi_{(1)}(p)$, which is given by Eq. (21), is momentum independent, it is natural to call it the *effective mass* and denote it as m_{eff}. So, the resumed Green's function equals

$$\Delta(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m_{\mathrm{eff}}^2} \tag{30}$$

and

$$m_{\mathrm{eff}}^2 \equiv \frac{\lambda}{24}T^2. \tag{31}$$

- Using the resumed propagator (30) instead of the free one, one has to reorganize the perturbative expansion (because m_{eff} depends on λ). To do it systematically, we write the Lagrangian density of the massless field as

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m_{\mathrm{eff}}^2\phi^2(x)}_{=\mathcal{L}_0}\underbrace{-\frac{\lambda}{4!}\phi^4(x) + \frac{1}{2}m_{\mathrm{eff}}^2\phi^2(x)}_{=\mathcal{L}_I}. \tag{32}$$

The term with the effective mass enters the free Lagrangian $\mathcal{L}_0$ and it is canceled by the same term of opposite sign in the interaction Lagrangian $\mathcal{L}_I$. So, the Lagrangian is actually unchanged.

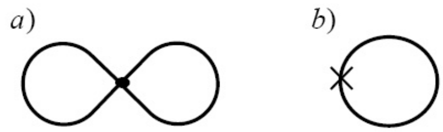

Figure 4: The first order contributions to $\ln Z$ in reorganized perturbative expansion

The first order

- The free partition must be now computed not for the massless but for massive field. As we remember, the partition function of non-interacting bosons of mass $m_{\rm eff}$ is

$$\ln Z_0 = -\frac{V}{2\pi^2}\int_0^\infty dk\, k^2 \ln(1-e^{-\beta\sqrt{m_{\rm eff}^2+k^2}}) = \frac{V}{6\pi^2 T}\int_0^\infty \frac{dk\, k^4}{\sqrt{m_{\rm eff}^2+k^2}}\frac{1}{e^{\beta\sqrt{m_{\rm eff}^2+k^2}}-1}. \quad (33)$$

- There is no closed analytical formula of the partition function (33) but keeping in mind that $m_{\rm eff} \ll T$ we can get an approximate result. The first two terms of the Taylor expansion around $m_{\rm eff}=0$ are

$$\ln Z_0 = \ln Z_0\Big|_{m_{\rm eff}^2=0} + \frac{\partial \ln Z_0}{\partial m_{\rm eff}^2}\Big|_{m_{\rm eff}^2=0} m_{\rm eff}^2 = \frac{\pi^2 V T^3}{90} - \frac{VTm_{\rm eff}^2}{24}. \quad (34)$$

  So, the free partition function of bosons produces the first order contribution because $m_{\rm eff}^2 \sim \lambda$.

- It is interesting to note that one cannot get the next term of the expansion (34) computing the second derivative of $\ln Z_0$ because the derivative produces a badly divergent integral. This signals that $\ln Z_0$ is not an analytic function of $m_{\rm eff}$. Consequently, the Taylor expansion (34) is not really reliable. However, the result (34) is correct. We will return to the issue.

- There are two first order contributions to $\ln Z$ of the reorganized perturbative expansion which correspond to the diagrams shown in Fig. 4. The right diagram comes from the extra mass term of the modified interaction Lagrangian and the cross in the diagram represents $m_{\rm eff}^2$.

- The diagram a), which has been already discussed, should be computed not with the free Green's function but with the resummed one (30). However, the contribution is, as we remember, infrared save in the massless limit and consequently the effect of effective mass is of higher order (actually of order λ^2). Therefore, we can use the already obtained result which is

$$\ln Z\Big|_{(1a)} = -\frac{\lambda}{4!}\,3\int_0^\beta d^4x\,\big(\Delta(x=0)\big)^2 = -\frac{\lambda}{8}\,\beta V\big(\Delta(x=0)\big)^2 = -\lambda\,\frac{VT^3}{1152}. \quad (35)$$

- As already mentioned, the diagram b) in Fig. 4 is due to the extra mass term in the interaction Lagrangian (32). Since the cross represents $m_{\rm eff}^2$, the contribution equals

$$\ln Z\Big|_{(1b)} = \frac{m_{\rm eff}^2}{2}\int_0^\beta d^4x\,\Delta(x=0) = \frac{VTm_{\rm eff}^2}{24}. \quad (36)$$

- One observes that the term (36) exactly cancels the contribution to $\ln Z_0$ given by Eq. (34) which is due to the effective mass. So, we have reproduced the known first order result with the reorganized perturbative expansion. Now, we are ready to deal with higher order contributions.

The next order

- The first contribution of order $\lambda^{3/2}$ comes from the free partition function and it is

$$\ln Z_0\Big|_{(3/2)} = \frac{V m_{\rm eff}^3}{12\pi} = \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{12\pi}. \tag{37}$$

- Somewhat unexpectedly it is truly difficult to obtain the expansion of $\ln Z_0$ of massive bosons around vanishing mass. The derivation of the term (37) can be found in the Appendix C of the classical paper by L. Dolan & R. Jackiw, Phys. Rev. D **9**, 3320 (1974). The idea of the derivation is to start not with $\ln Z_0$ given by Eq. (33) but the modified formula which includes an extra factor $k^{-\epsilon}$. The factor guarantees that the higher order derivatives of $\ln Z_0$ with respect to m^2 are convergent integrals. In the final step one takes the limit $\epsilon \to 0$.

- We note that the contribution (37) exactly equals the sum of ring diagrams (27). So, one expects that the remaining contributions of order $\lambda^{3/2}$ cancel each other.

- The second contribution of order $\lambda^{3/2}$ appears as a subleading part of the first order diagram a) shown in Fig. 4 when computed with the resummed propagator (30). The diagram provides

$$\ln Z\Big|_{(1a)} = -\frac{\lambda}{4!}\,3\int_0^\beta d^4x\,\big(\Delta(x=0)\big)^2 = -\frac{\lambda}{8}\,\beta V\big(\Delta(x=0)\big)^2, \tag{38}$$

where

$$\Delta(x=0) = T\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_n^2+\mathbf{p}^2+m_{\rm eff}^2}. \tag{39}$$

- The subleading contribution of interest is due to zero Matsubara frequency. For this reason we express the propagator $\Delta(x=0)$ as

$$\Delta(x=0) = T\sum_{n=-\infty,\;n\neq 0}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_n^2+\mathbf{p}^2+m_{\rm eff}^2} + T\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\mathbf{p}^2+m_{\rm eff}^2}. \tag{40}$$

Since $m_{\rm eff} \ll \omega_n$ if $n \neq 0$, we can neglect $m_{\rm eff}$ in the first term of Eq. (40) which we rewrite as follows

$$\begin{aligned}\Delta(x=0) &= T\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_n^2+\mathbf{p}^2} - T\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\mathbf{p}^2} + T\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\mathbf{p}^2+m_{\rm eff}^2}\\ &= T\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_n^2+\mathbf{p}^2} + T\int\frac{d^3p}{(2\pi)^3}\Big[\frac{1}{\mathbf{p}^2+m_{\rm eff}^2}-\frac{1}{\mathbf{p}^2}\Big]. \end{aligned} \tag{41}$$

The first term has been already computed and it equals $\frac{T^2}{12}$ while the second term is computed as

$$\frac{T}{2\pi^2}\int_0^\infty dp\,p^2\Big[\frac{1}{p^2+m_{\rm eff}^2}-\frac{1}{p^2}\Big] = -\frac{m_{\rm eff}^2T}{2\pi^2}\int_0^\infty\frac{dp}{p^2+m_{\rm eff}^2} = -\frac{m_{\rm eff}T}{2\pi^2}\underbrace{\int_0^\infty\frac{dx}{x^2+1}}_{=\frac{\pi}{2}} = -\frac{m_{\rm eff}T}{4\pi}.$$

One finally obtains

$$\Delta(x=0) = \frac{T^2}{12} - \frac{m_{\rm eff}T}{4\pi}. \tag{42}$$

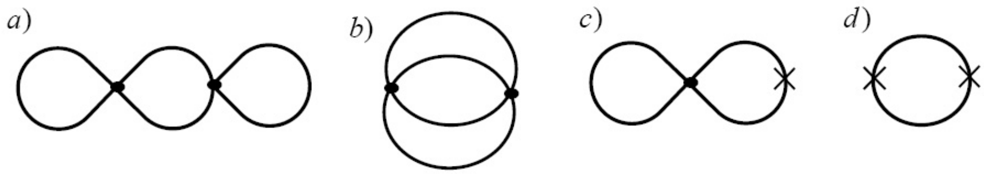

Figure 5: The second order contributions to $\ln Z$ in reorganized perturbative expansion

- Substituting the result (42) into Eq. (38), we get

$$\ln Z\Big|_{(1a)} = -\frac{\lambda}{8}\,\beta V\Big(\frac{T^2}{12} - \frac{m_{\rm eff}T}{4\pi}\Big)^2, \tag{43}$$

and the subleading contribution of order $\lambda^{3/2}$ equals

$$\ln Z\Big|_{(1a)}^{\rm sub} = \frac{\lambda}{8}\,\frac{V m_{\rm eff}T^3}{24\pi} = \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{8\pi} \tag{44}$$

- <u>The third contribution of order $\lambda^{3/2}$</u> is a subleading part of the first order diagram b) shown in Fig. 4 computed with the resummed propagator (30). The diagram, which is due to the extra mass term in the interaction Lagrangian (32), provides

$$\ln Z\Big|_{(1b)} = \frac{m_{\rm eff}^2}{2}\int_0^\beta d^4x\,\Delta(x=0) = \frac{m_{\rm eff}^2}{2}\,\beta V\,\Delta(x=0). \tag{45}$$

Using the formula (42), one immediately finds the subleading contribution as

$$\ln Z\Big|_{(1b)}^{\rm sub} = -\frac{V m_{\rm eff}^3}{8\pi} = -\Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{8\pi}. \tag{46}$$

- One sees that the contributions (44) and (46) cancel each other exactly.
- There are four second order contributions to $\ln Z$ of the reorganized perturbative expansion which correspond to the diagrams shown in Fig. 5.
- <u>The fourth contribution of order $\lambda^{3/2}$</u> is provided by the second order diagram a) from Fig. 1 or Fig. 5 when computed with the resummed Green's function (30). So, we return to Eq. (8) where we substitute the propagator (30). Thus, we get

$$I = \frac{V}{2\pi^2}\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{dp\,p^2}{(\omega_n^2+p^2+m_{\rm eff}^2)^2}. \tag{47}$$

- Since the contribution of interest is due to the zero Matsubara frequency, Eq. (47) gives

$$I_{n=0} = \frac{V}{2\pi^2}\int_0^\infty \frac{dp\,p^2}{(m_{\rm eff}^2+p^2)^2} = \frac{V}{2\pi^2 m_{\rm eff}}\underbrace{\int_0^\infty \frac{dx\,x^2}{(1+x^2)^2}}_{=\frac{\pi}{4}} = \frac{V}{8\pi\,m_{\rm eff}}. \tag{48}$$

- Plugging the result (48) into Eq. (2), one obtains

$$\ln Z\Big|_{(2a)} = \frac{\lambda^2}{16}\,\frac{T^4}{144}\,\frac{V}{8\pi\,m_{\rm eff}} = \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{32\pi}. \tag{49}$$

- As we remember, the diagram b) in Fig. 1 or Fig. 5 is infrared save when computed with free massless propagator. So, it does not produce the contribution of order $\lambda^{3/2}$
- <u>The fifth contribution of order $\lambda^{3/2}$</u> corresponds to the diagram c) in Fig. 5. It equals

$$\ln Z\Big|_{(2c)} = -\frac{\lambda m_{\rm eff}^2}{2\cdot 4!}\, 3\cdot 4 \int_0^\beta d^4x \int_0^\beta d^4y\, \Delta(x=0)\,\Delta(x-y)\,\Delta(y-x)$$
$$= -\frac{\lambda m_{\rm eff}^2}{4}\,\Delta(x=0)\, I, \tag{50}$$

where the quantity I is defined by Eq. (3) and it is given by the formula (48). Therefore,

$$\ln Z\Big|_{(2c)} = -\frac{\lambda m_{\rm eff} V T^2}{384\pi} = -\Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{16\pi}. \tag{51}$$

- <u>The sixth contribution of order $\lambda^{3/2}$</u> corresponds to the diagram d) in Fig. 5. It equals

$$\ln Z\Big|_{(2d)} = \frac{m_{\rm eff}^4}{4}\int_0^\beta d^4x \int_0^\beta d^4y\, \Delta(x-y)\,\Delta(y-x)$$
$$= \frac{m_{\rm eff}^4}{4}\, I = \frac{m_{\rm eff}^3 V}{32\pi} = \Big(\frac{\lambda}{4!}\Big)^{3/2}\frac{VT^3}{32\pi}. \tag{52}$$

- One sees that the contributions (49, 51) and (52) cancel each other exactly. So, we conclude that with the reorganized perturbative expansion we have reproduced the results given by Eq. (28).
- An advantage of the reorganized perturbative expansion becomes evident when one goes to higher orders.