Real-time or Keldysh-Schwinger formalism

As already mentioned, there are two formulations of statistical QFT: the *imaginary-time* or *Matsubara formalism*, which have been already introduced, and the *real-time* or *Keldysh-Schwinger formalism.* Now, we are going to introduce the latter one which is applicable to both equilibrium and non-equilibrium systems. The real-time formalism was foreshadowed by the work of Julian Schwinger (1918-1994) and proposed almost simultaneously by Leonid Keldysh (1931-2016), and independently by Leo Kadanoff (1937-2015) and Gordon Baym (born 1935).

Green's functions

Definitions

- The key object of the Keldysh-Schwinger formalism is the contour Green's function which for a complex scalar field represented by the operator $\hat{\phi}(x)$ is defined as

$$i\Delta(x,y) \equiv \langle \tilde{T}\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\rangle. \tag{1}$$

As previously, the hats are used to denote operators acting in the space of states. The operation $\langle \ldots \rangle$ is understood as

$$\langle \ldots \rangle \equiv \frac{\mathrm{Tr}[\hat{\rho}(t_0)\ldots]}{\mathrm{Tr}[\hat{\rho}(t_0)]}, \tag{2}$$

where $\hat{\rho}(t_0)$ is a density operator and the trace is a sum over a complete set of states of the system at a given initial time t_0,

$$\mathrm{Tr}[\ldots] \equiv \sum_{\alpha} \langle \alpha|\ldots|\alpha\rangle. \tag{3}$$

The symbol $\tilde{T}$ represents the operation of ordering along the Keldysh contour shown in Fig. 1. The time arguments of the fileds are complex with an infinitesimal positive or negative imaginary part which locates them on the upper or lower branch of the contour. The contour ordering operation of two arbitrary operators $\hat{A}(x)$ and $\hat{B}(y)$ is defined as

$$\tilde{T}\hat{A}(x)\hat{B}(y) \equiv \Theta_C(x_0,y_0)\hat{A}(x)\hat{B}(y) \pm \Theta_C(y_0,x_0)\hat{B}(y)\hat{A}(x), \tag{4}$$

where $\Theta_C(x_0,y_0)$ is the contour step function defined as

$$\Theta_C(x_0,y_0) = \begin{cases} 1 & \text{if } x_0 \text{ succeeds } y_0 \text{ along the contour,} \\ 0 & \text{if } y_0 \text{ succeeds } x_0 \text{ along the contour.} \end{cases} \tag{5}$$

The plus sign in the formula (4) is for bosonic operators whereas the minus sign is for fermionic operators. The parameter $t_{\max}$, which is bigger that x_0 and y_0, is usually shifted to $+\infty$ and t_0, which is smaller that x_0 and y_0, is shifted to $-\infty$ in calculations.

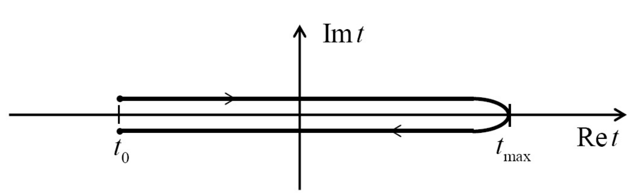

Figure 1: The Keldysh contour

- The contour function (1) represents four Green's functions of real-time arguments which are

$$\begin{aligned} i\Delta^{>}(x,y) &\equiv \langle \hat{\phi}(x)\hat{\phi}^{\dagger}(y)\rangle, & (6)\\ i\Delta^{<}(x,y) &\equiv \langle \hat{\phi}^{\dagger}(y)\hat{\phi}(x)\rangle, & (7)\\ i\Delta^{c}(x,y) &\equiv \langle T^{c}\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\rangle, & (8)\\ i\Delta^{a}(x,y) &\equiv \langle T^{a}\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\rangle, & (9)\end{aligned}$$

  where T^c is a chronological time ordering defined as

$$T^{c}\hat{A}(x)\hat{B}(y) \equiv \Theta(x_0 - y_0)\hat{A}(x)\hat{B}(y) \pm \Theta(y_0 - x_0)\hat{B}(y)\hat{A}(x), \tag{10}$$

  and T^a is an antichronological time ordering

$$T^{a}\hat{A}(x)\hat{B}(y) \equiv \Theta(y_0 - x_0)\hat{A}(x)\hat{B}(y) \pm \Theta(x_0 - y_0)\hat{B}(y)\hat{A}(x). \tag{11}$$

  The plus sign is for bosonic operators and the minus for fermionic ones.

- The contour Green's function (1)) is related the real-time Green's functions (6, 7, 8, 9) as

$$\Delta(x,y) = \begin{cases} \Delta^{>}(x,y) & \text{for } x_0 \text{ on the lower branch and } y_0 \text{ on the upper one,}\\ \Delta^{<}(x,y) & \text{for } x_0 \text{ on the upper branch and } y_0 \text{ on the lower one,}\\ \Delta^{c}(x,y) & \text{for } x_0 \text{ and } y_0 \text{ on the upper branch,}\\ \Delta^{a}(x,y) & \text{for } x_0 \text{ and } y_0 \text{ on the lower branch.}\end{cases}$$

- As we will discuss in detail soon, the functions $\Delta^{>}$ and $\Delta^{<}$ play a role of the phase-space density of (quasi-)particles, and they can be treated as quantum analogs of classical distribution functions. The function Δ^c describes a particle propagating forward in time, and an antiparticle propagating backward in time. The meaning of Δ^a is analogous but particles are propagated backward in time and antiparticles forward. In the zero density limit Δ^c coincides with the usual Feynman propagator.

- The functions of real-time arguments are often assembled in a 2×2 matrix which is written as

$$\begin{pmatrix} \Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix} = \begin{pmatrix} \Delta^{c} & \Delta^{>}\\ \Delta^{<} & \Delta^{a}\end{pmatrix}. \tag{12}$$

  As seen, the matrix elements with the indices $i, j = 1$ correspond to functions of time arguments located on the upper branch of the Keldysh contour and these indexed by $i, j = 2$ refer to lower branch of the time contour.

- It is often needed to work with retarded (+), advanced (−) and symmetric Green's functions defined as

$$\begin{aligned} i\Delta^{+}(x,y) &\equiv \Theta(x_0 - y_0)\langle[\hat{\phi}(x), \hat{\phi}^{\dagger}(y)]\rangle, & (13)\\ i\Delta^{-}(x,y) &\equiv -\Theta(y_0 - x_0)\langle[\hat{\phi}(x), \hat{\phi}^{\dagger}(y)]\rangle, & (14)\\ i\Delta^{\rm sym}(x,y) &\equiv \langle\{\hat{\phi}(x), \hat{\phi}^{\dagger}(y)\}\rangle, & (15)\end{aligned}$$

  where $[\ldots,\ldots]$ denotes a commutator and $\{\ldots,\ldots\}$ an anticommutator of operators. The retarded (avanced) Green's function describes the propagation of both particle and antiparticle disturbance forward (backward) in time.

- Finally, we define the spectral function as

$$A(x,y) \equiv \langle [\hat{\phi}(x), \hat{\phi}^\dagger(y)] \rangle = i\Big(\Delta^>(x,y) - \Delta^<(x,y)\Big). \tag{16}$$

The function provides information about a spectrum of excitations of a system that is what types of (quasi-)particles we deal with.

Relations among Green's functions

- First of all, we note that the four Green's function of real-time arguments (6, 7, 8, 9) are not independent from each other. Directly from the definitions, one shows that

$$\Delta^{\substack{c\\a}}(x,y) = \Theta(x_0 - y_0)\Delta^{\gtrless}(x,y) + \Theta(y_0 - x_0)\Delta^{\lessgtr}(x,y), \tag{17}$$

which leads to

$$\Delta^c(x,y) + \Delta^a(x,y) = \Delta^>(x,y) + \Delta^<(x,y). \tag{18}$$

- Analogously, we have

$$\begin{aligned} \Delta^+(x,y) - \Delta^-(x,y) &= \Delta^>(x,y) - \Delta^<(x,y), && (19)\\ \Delta^{\rm sym}(x,y) &= \Delta^>(x,y) + \Delta^<(x,y). && (20) \end{aligned}$$

- Treating the Green's functions as matrices, one finds the relations

$$\begin{aligned} \Big(i\Delta^{\substack{c\\a}}(x,y)\Big)^\dagger &= i\Delta^{\substack{a\\c}}(x,y), && (21)\\ \Big(i\Delta^{\gtrless}(x,y)\Big)^\dagger &= i\Delta^{\gtrless}(x,y), && (22) \end{aligned}$$

where † means the Hermitian conjugation which involves the complex conjugation and interchange of the arguments of the functions.

- When the field $\phi(x)$ is real, there are also specific relations

$$\begin{aligned} \Delta^{\lessgtr}(x,y) &= \Delta^{\gtrless}(y,x), && (23)\\ \Delta^{\substack{a\\c}}(x,y) &= \Delta^{\substack{c\\a}}(y,x). && (24) \end{aligned}$$

Physical meaning of $\boldsymbol{\Delta}^{\gtrless}$

- The functions $\Delta^{\substack{c\\a}}$ and $\Delta^\pm$ are commonly used in vacuum QFT and their physical meaning in the statistical QFT is similar. The case of unordered functions $\Delta^{\gtrless}$, which play an important role the statistical field theory, is different. So, let us discuss a physical interpretation of the functions in some detail.

- We consider a free massive complex scalar field. The Lagrangian density of classical theory is

$$\mathcal{L}(x) = \partial^\mu\phi(x)\partial_\mu\phi^\dagger(x) - m^2\phi(x)\phi^\dagger(x). \tag{25}$$

In such a theory, as well known, the current vector and energy-momentum tensor, which are defined as

$$\begin{aligned} j^\mu(x) &\equiv i\phi(x)\partial^\mu\phi^\dagger(x) - i\big(\partial^\mu\phi(x)\big)\phi^\dagger(x), && (26)\\ T^{\mu\nu}(x) &\equiv \big(\partial^\mu\phi(x)\big)\big(\partial^\nu\phi^\dagger(x)\big) + \big(\partial^\nu\phi(x)\big)\big(\partial^\mu\phi^\dagger(x)\big) - g^{\mu\nu}\mathcal{L}(x), && (27) \end{aligned}$$

obey the continuity equations

$$\partial_\mu j^\mu(x) = 0, \qquad \partial_\mu T^{\mu\nu}(x) = 0. \tag{28}$$

- We modify the energy-momentum tensor subtracting from the expression (27) the total derivative

$$\frac{1}{2}\partial^\mu\partial^\nu\big(\phi(x)\phi^\dagger(x)\big) - \frac{1}{4}g^{\mu\nu}\partial^\sigma\partial_\sigma\big(\phi(x)\phi^\dagger(x)\big). \tag{29}$$

Then, the energy-momentum tensor, which still obeys the continuity equation, becomes

$$T^{\mu\nu}(x) = -\frac{1}{2}\phi(x)\overleftrightarrow{\partial^\mu}\overleftrightarrow{\partial^\nu}\phi^\dagger(x), \tag{30}$$

where the fields are assumed to satisfy the Klein-Gordon equations and the derivative $\overleftrightarrow{\partial^\mu}$ acting on functions A and B is understood as

$$A\overleftrightarrow{\partial_\mu}B = A\partial_\mu B - \big(\partial_\mu A\big)B. \tag{31}$$

- We are going to express the quantities

$$\begin{aligned}\big\langle \hat{j}^\mu(x)\big\rangle &= i\big\langle\hat{\phi}(x)\overleftrightarrow{\partial^\mu}\hat{\phi}^\dagger(x)\big\rangle, &(32)\\ \big\langle \hat{T}^{\mu\nu}(x)\big\rangle &= -\frac{1}{2}\big\langle\hat{\phi}(x)\overleftrightarrow{\partial^\mu}\overleftrightarrow{\partial^\nu}\hat{\phi}^\dagger(x)\big\rangle, &(33)\end{aligned}$$

through the functions $\Delta^{\gtrless}$ and we observe that

$$\begin{aligned}\big\langle \hat{j}^\mu(x)\big\rangle &= \big(\partial_x^\mu - \partial_y^\mu\big)\Delta^>(x,y)\Big|_{x=y}, &(34)\\ \big\langle \hat{T}^{\mu\nu}(x)\big\rangle &= -\frac{i}{2}\Big(\partial_y^\mu\partial_y^\nu - \partial_x^\mu\partial_y^\nu - \partial_x^\nu\partial_y^\nu + \partial_x^\mu\partial_y^\nu\Big)\Delta^>(x,y)\Big|_{x=y}. &(35)\end{aligned}$$

- Introducing the variables

$$X \equiv \frac{1}{2}(x+y), \qquad u \equiv x - y, \tag{36}$$

the Green's function is written as

$$\Delta^>(x,y) = \Delta^>\big(X + \frac{1}{2}u, X - \frac{1}{2}u\big) \equiv \Delta^>(X,u). \tag{37}$$

- Keeping in mind that

$$\begin{aligned}\partial_x^\mu - \partial_y^\mu &= 2\partial_u^\mu, &(38)\\ \partial_y^\mu\partial_y^\nu - \partial_x^\mu\partial_y^\nu - \partial_x^\nu\partial_y^\nu + \partial_x^\mu\partial_y^\nu &= 4\partial_u^\mu\partial_u^\nu, &(39)\end{aligned}$$

the equations (34, 35) are expressed as

$$\begin{aligned}\big\langle \hat{j}^\mu(X)\big\rangle &= 2\partial_u^\mu\Delta^>(X,u)\Big|_{u=0}, &(40)\\ \big\langle \hat{T}^{\mu\nu}(X)\big\rangle &= -2i\partial_u^\mu\partial_u^\nu\Delta^>(X,u)\Big|_{u=0}. &(41)\end{aligned}$$

- Now, we introduce the Wigner transformation

$$\Delta^{>}(X,p) = \int d^4u \, e^{ipu} \, \Delta^{>}(X,u), \tag{42}$$

which is the Fourier transformation in the variable u. The inverse Wigner transform is

$$\Delta^{>}(X,u) = \int \frac{d^4p}{(2\pi)^4} e^{-ipu} \Delta^{>}(X,p). \tag{43}$$

- Using the Wigner transformed function $\Delta^{>}(X,p)$, the equations (40, 41) read

$$\langle \hat{j}^{\mu}(X) \rangle = 2\partial_u^{\mu} \int \frac{d^4p}{(2\pi)^4} e^{-ipu} \Delta^{>}(X,p) \Big|_{u=0}, \tag{44}$$

$$\langle \hat{T}^{\mu\nu}(X) \rangle = -2i \, \partial_u^{\mu}\partial_u^{\nu} \int \frac{d^4p}{(2\pi)^4} e^{-ipu} \Delta^{>}(X,p) \Big|_{u=0}. \tag{45}$$

where one sets $u = 0$ after the differentiation over u.

- Thus, we finally obtain

$$\langle \hat{j}^{\mu}(X) \rangle = -2 \int \frac{d^4p}{(2\pi)^4} p^{\mu} \, i\Delta^{>}(X,p), \tag{46}$$

$$\langle \hat{T}^{\mu\nu}(X) \rangle = 2 \int \frac{d^4p}{(2\pi)^4} p^{\mu}p^{\nu} \, i\Delta^{>}(X,p). \tag{47}$$

- One observes that exactly the same result is found with the function $\Delta^{<}$.

- The formulas (46, 47) show that the functions $i\Delta^{\gtrless}(X,p)$ give the phase-space density of particles that is they are quantum analogs of classical distribution functions. This interpretation is supported by the fact that $i\Delta^{>}(X,p)$ and $i\Delta^{<}(X,p)$ are both Hermitian, however, they are not positive definite and thus the probabilistic interpretation is only approximately valid. One should also observe that, in contrast to the classical distribution functions, $i\Delta^{\gtrless}(X,p)$ can be nonzero for the off-mass-shell four-momenta, when $p^2 \neq m^2$. As we will see soon, the functions $i\Delta^{\gtrless}(X,p)$ vanish for off-mass-shell momenta under some conditions.

Free Green's functions of equilibrium system

One often needs an explicit form of Green's functions. So, we derive here the free functions of equilibrium system. The derived formulas allow one to better understand a physical meaning of the Green's functions.

- The density matrix of equilibrium system, which is written in the heat bath rest frame, equals

$$\hat{\rho} = e^{-\beta \hat{H}}, \tag{48}$$

where $\beta \equiv T^{-1}$ is the inverse temperature and $\hat{H}$ is the system's Hamiltonian.

- The equilibrium contour Green's function (1) is

$$i\Delta(x-y) = \frac{1}{Z}\mathrm{Tr}[e^{-\beta\hat{H}}\tilde{T}\hat{\phi}(x)\hat{\phi}^\dagger(y)], \tag{49}$$

where $Z(T) \equiv \mathrm{Tr}[e^{-\beta\hat{H}}]$ is the partition function, see Eq. (7) of Lecture II. In case of non-interacting fields the function was already computed, see Eq. (19) of Lecture II.

- Equilibrium systems are homogeneous or translationally invariant in absence of external forces. Consequently, the Green's functions depend on x and y only through $x-y$.

- We first compute the $\Delta^>(x,y)$, using the method presented in Lecture II.

- The discrete version of the plane-wave decomposition of the field operator

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(\hat{a}(\mathbf{k})e^{-ikx} + \hat{a}^\dagger(\mathbf{k})e^{ikx}\Big), \tag{50}$$

where $k^\mu = (\omega_\mathbf{k}, \mathbf{k})$ and $\omega_\mathbf{k} \equiv \sqrt{\mathbf{k}^2 + m^2}$, is

$$\hat{\phi}(x) = \sqrt{\Delta_\mathbf{k}}\sum_i \frac{1}{\sqrt{2\omega_i}}\Big(\hat{a}_i e^{-ik_ix} + \hat{a}_i^\dagger e^{ik_ix}\Big), \tag{51}$$

with $\Delta_\mathbf{k}$ being the volume of a momentum-space cell, $\hat{a}_i \equiv \sqrt{\Delta_\mathbf{k}}\hat{a}(\mathbf{k}_i)$ and $\hat{a}_i^\dagger \equiv \sqrt{\Delta_\mathbf{k}}\hat{a}^\dagger(\mathbf{k}_i)$.

- The discrete operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are dimensionless and satisfy the following commutation relations

$$\big[\hat{a}_i, \hat{a}_j^\dagger\big] = \delta^{ij}, \qquad \big[\hat{a}_i, \hat{a}_j\big] = 0, \qquad \big[\hat{a}_i^\dagger, \hat{a}_j^\dagger\big] = 0. \tag{52}$$

- The discrete normally ordered Hamiltonian is

$$\hat{H} = \sum_i \omega_i \hat{a}_i^\dagger \hat{a}_i = \sum_i \omega_i \hat{n}_i, \tag{53}$$

where $\hat{n}_i \equiv \hat{a}_i^\dagger\hat{a}_i$.

- As discussed in Lecture I, the Fock space is built of the mutually orthogonal energy eigenstates $|n_1, n_2, n_3, \ldots\rangle$, and the action of the annihilation and creation operators is defined in the following way

$$\hat{a}_i|n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, n_2, \ldots, n_i - 1, \ldots\rangle, \tag{54}$$

$$\hat{a}_i^\dagger|n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\,|n_1, n_2, \ldots, n_i + 1, \ldots\rangle. \tag{55}$$

- Keeping in mind that the Green's function $\Delta^>(x,y)$ equals

$$\begin{aligned}\Delta^>(x-y) &= Z^{-1}\int \frac{d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}\sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots \\ &\times\big\langle n_1, n_2, \ldots\big|\exp(-\beta\hat{H})\Big(\hat{a}(\mathbf{k})e^{-ikx} + \hat{a}^\dagger(\mathbf{k})e^{ikx}\Big) \\ &\times\Big(\hat{a}(\mathbf{p})e^{-ipy} + \hat{a}^\dagger(\mathbf{p})e^{ipy}\Big)\big|n_1, n_2, \ldots\big\rangle,\end{aligned} \tag{56}$$

the discrete version is

$$\Delta^>(x-y) = Z^{-1}\Delta_{\mathbf{k}}\sum_{n_1}\sum_{n_2}\sum_{n_3}\dots e^{-\beta\omega_1 n_1}e^{-\beta\omega_2 n_2}e^{-\beta\omega_3 n_3}\dots\sum_i\sum_j\frac{1}{2\sqrt{\omega_i\omega_j}} \tag{57}$$
$$\times\Big[e^{-i(k_i x+p_j y)}\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i\hat{a}_j\big|n_1,\dots,n_j,\dots\big\rangle$$
$$+e^{-i(k_i x-p_j y)}\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i\hat{a}_j^\dagger\big|n_1,\dots,n_j,\dots\big\rangle$$
$$+e^{i(k_i x-p_j y)}\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i^\dagger\hat{a}_j\big|n_1,\dots,n_j,\dots\big\rangle$$
$$+e^{i(k_i x+p_j y)}\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i^\dagger\hat{a}_j^\dagger\big|n_1,\dots,n_j,\dots\big\rangle\Big],$$

where $k_i^\mu = (\omega_i, \mathbf{k}_i)$ and $p_i^\mu = (\omega_i, \mathbf{p}_i)$.

- Computing

$$\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i\hat{a}_j\big|n_1,\dots,n_j,\dots\big\rangle = 0, \tag{58}$$
$$\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i^\dagger\hat{a}_j^\dagger\big|n_1,\dots,n_j,\dots\big\rangle = 0, \tag{59}$$
$$\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i\hat{a}_j^\dagger\big|n_1,\dots,n_j,\dots\big\rangle = \sqrt{(n_i+1)(n_j+1)}\;\delta^{ij}, \tag{60}$$
$$\big\langle n_1,\dots,n_i,\dots\big|\hat{a}_i^\dagger\hat{a}_j\big|n_1,\dots,n_j,\dots\big\rangle = \sqrt{n_i n_j}\;\delta^{ij}, \tag{61}$$

one finds

$$\Delta^>(x-y) = Z^{-1}\Delta_{\mathbf{k}}\sum_{n_1}\sum_{n_2}\sum_{n_3}\dots e^{-\beta\omega_1 n_1}e^{-\beta\omega_2 n_2}e^{-\beta\omega_3 n_3}\dots \tag{62}$$
$$\times\sum_i\frac{1}{2\omega_i}\Big[n_i\Big(e^{-ik_i(x-y)}+e^{ik_i(x-y)}\Big)+e^{-ik_i(x-y)}\Big].$$

- Using the formulas

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \qquad \sum_{n=0}^{\infty} n q^n = \frac{q}{(1-q)^2}, \tag{63}$$

one obtains

$$i\Delta^>(x) = \Delta_{\mathbf{k}}\sum_i\frac{1}{2\omega_i}\left[\frac{e^{-\beta\omega_i}}{1-e^{-\beta\omega_i}}\Big(e^{-ik_i x}+e^{ik_i x}\Big)+e^{-ik_i x}\right], \tag{64}$$

where the partition function Z (given by Eq. (17) of Lecture II) cancels out.

- In the final step we change a discrete momentum space into a continuous one and we get

$$i\Delta^>(x) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\Big[\big(f(\omega_{\mathbf{k}})+1\big)e^{-ikx}+f(\omega_{\mathbf{k}})e^{ikx}\Big], \tag{65}$$

where the boson distribution function equals

$$f(\omega_{\mathbf{k}}) \equiv \frac{1}{e^{\beta\omega_{\mathbf{k}}}-1}. \tag{66}$$

- Repeating the same steps as in case of $\Delta^>$, we find $\Delta^<$ as

$$i\Delta^<(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\Big[f(\omega_{\mathbf{k}})e^{-ikx} + \big(f(\omega_{\mathbf{k}})+1\big)e^{ikx}\Big]. \tag{67}$$

Exercise: Derive the formula (67).

- Performing the Fourier transformation and keeping in mind that

$$\int dt\, d^3x\; e^{i(k-p)x} = (2\pi)^4\delta(\omega_{\mathbf{k}} - p_0)\delta^{(3)}(\mathbf{k}-\mathbf{p}), \tag{68}$$

one obtains the unordered functions $\Delta^{\gtrless}$ in the momentum space as

$$\boxed{\Delta^>(p) = -\frac{i\pi}{\omega_{\mathbf{p}}}\Big[\delta(\omega_{\mathbf{p}} - p_0)(f(\omega_{\mathbf{p}})+1) + \delta(\omega_{\mathbf{p}}+p_0)f(\omega_{\mathbf{p}})\Big],} \tag{69}$$

$$\boxed{\Delta^<(p) = -\frac{i\pi}{\omega_{\mathbf{p}}}\Big[\delta(\omega_{\mathbf{p}} - p_0)f(\omega_{\mathbf{p}}) + \delta(\omega_{\mathbf{p}}+p_0)(f(\omega_{\mathbf{p}})+1)\Big].} \tag{70}$$

We see that the functions $\Delta^>(p)$ and $\Delta^<(p)$ are non zero only for $p^2 = m^2$.

Exercise: Compute the current (46) and energy-momentum tensor (47) using the function (69).

- The functions $\Delta^>(p)$ and $\Delta^<(p)$ obey the Kubo-Martin-Schwinger boundary condition (KMS condition)

$$\boxed{\Delta^<(p) = e^{-\beta p_0}\Delta^>(p),} \tag{71}$$

which is derived at the end of the lecture.

- Once we know the functions $\Delta^>(p)$ and $\Delta^<(p)$, we immediately get the spectral function, which is defined by Eq. (16), as

$$A(p) = i\Big(\Delta^>(p) - \Delta^<(p)\Big) = \frac{\pi}{\omega_{\mathbf{p}}}\Big[\delta(\omega_{\mathbf{p}} - p_0) - \delta(\omega_{\mathbf{p}} + p_0)\Big]. \tag{72}$$

We note that the spectral function (72) holds for equilibrium and non-equilibrium non-interacting systems. In case of free field it can obtained directly from the definition (16) using the plane-wave decomposition of the fields and the commutation relation of creation and annihilation operators.

- Let us now compute the Green functions Δ^c and Δ^a. Due to the relation (17), we have

$$\begin{aligned}\Delta^c(x) \;&=\; -\frac{i}{2}\bigg\{\Theta(x_0)\int\frac{d^3k}{(2\pi)^3\omega_{\mathbf{k}}}\Big[\big(f(\omega_{\mathbf{k}})+1\big)e^{-ikx} + f(\omega_{\mathbf{k}})e^{ikx}\big)\Big] \\ &\quad +\Theta(-x_0)\int\frac{d^3k}{(2\pi)^3\omega_{\mathbf{k}}}\Big[f(\omega_{\mathbf{k}})e^{-ikx} + \big(f(\omega_{\mathbf{k}})+1\big)e^{ikx}\big)\Big]\bigg\}.\end{aligned} \tag{73}$$

- Performing the Fourier transformation, we obtain

$$\begin{aligned}\Delta^c(p) \;&=\; -\frac{i}{2\omega_{\mathbf{p}}}\bigg\{\int_{-\infty}^{\infty} dt\; f(\omega_{\mathbf{p}})\Big[e^{i(p_0-\omega_{\mathbf{p}})t} + e^{i(p_0+\omega_{\mathbf{p}})t}\Big] \\ &\quad + \int_0^{\infty} dt\; e^{i(p_0-\omega_{\mathbf{p}})t} + \int_{-\infty}^{0} dt\; e^{i(p_0+\omega_{\mathbf{p}})t}\bigg\}.\end{aligned} \tag{74}$$

- Since the second and third terms in Eq. (74) are ill defined, we replace $\omega_{\mathbf{p}} \to \omega_{\mathbf{p}} \mp i0^+$ to make the limit $t \to \pm\infty$ meaningful. Then, we have

$$\int_0^\infty dt\; e^{i(p_0-\omega_{\mathbf{p}}+i0^+)t} = \frac{-ie^{i(p_0-\omega_{\mathbf{p}}+i0^+)t}}{p_0-\omega_{\mathbf{p}}+i0^+}\Big|_0^\infty = \frac{i}{p_0-\omega_{\mathbf{p}}+i0^+} \tag{75}$$

and

$$\int_{-\infty}^0 dt\; e^{i(p_0+\omega_{\mathbf{p}}-i0^+)t} = \frac{-ie^{i(p_0+\omega_{\mathbf{p}}-i0^+)t}}{p_0+\omega_{\mathbf{p}}-i0^+}\Big|_{-\infty}^0 = \frac{-i}{p_0+\omega_{\mathbf{p}}-i0^+}. \tag{76}$$

Thus, one obtains

$$\boxed{\Delta^c(p) = \frac{1}{p^2-m^2+i0^+} - \frac{i\pi}{\omega_{\mathbf{p}}} f(\omega_{\mathbf{p}}) \Big[\delta(p_0-\omega_{\mathbf{p}}) + \delta(p_0+\omega_{\mathbf{p}})\Big],} \tag{77}$$

and analogously

$$\boxed{\Delta^a(p) = -\frac{1}{p^2-m^2-i0^+} - \frac{i\pi}{\omega_{\mathbf{p}}} f(\omega_{\mathbf{p}}) \Big[\delta(p_0-\omega_{\mathbf{p}}) + \delta(p_0+\omega_{\mathbf{p}})\Big].} \tag{78}$$

One recognizes the first term of Eq. (77) as the usual Feynman propagator. The second term represents the effect of a medium. The first term is non-zero for any p while the second one only for $p^2 = m^2$.

- We see that the Feynman propagator in statistical quantum field theory (77) differs from its counterpart in the vacuum theory. The difference makes the second term in Eq. (77). Instead of a propagation of a particle (antiparticle) from the space-time point x to the space-time y, the particle (antiparticle) can absorbed by a medium at x and another identical particle (antiparticle) of the same four-momentum can be emitted at y.

<u>Exercise</u>: Derive the formula (78).

- The retarded and advanced Green's functions Δ^+ and Δ^- can be found using the relations

$$\Delta^+(p) = \Delta^c(p) - \Delta^<(p), \qquad \Delta^-(p) = \Delta^c(p) - \Delta^>(p). \tag{79}$$

- The formulas (77) and (70) provide

$$\Delta^+(p) = \frac{i\pi}{\omega_{\mathbf{p}}}\delta(p_0+\omega_{\mathbf{p}}) + \frac{1}{p^2-m^2+i0^+}. \tag{80}$$

which can be rewritten as

$$\boxed{\Delta^+(p) = \frac{1}{p^2-m^2+i\mathrm{sgn}(p_0)0^+},} \tag{81}$$

using the identity

$$\frac{1}{x \pm i0^+} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x). \tag{82}$$

- Analogously one finds

$$\boxed{\Delta^-(p) = \frac{1}{p^2-m^2-i\mathrm{sgn}(p_0)0^+}.} \tag{83}$$

- We observe that the propagators $\Delta^{\pm}(p)$ given by Eqs. (81) and (83) coincide with those of a vacuum QFT, that is, there is no medium contribution – the distribution function $f(\omega_{\mathbf{p}})$ does not show up in Eqs. (81) and (83).

Free Green's functions of non-equilibrium system

The non-equilibrium Green's functions of non-interacting fields can be found by solving appropriate equations of motion.

- Using the Klein-Gordon equations

$$\left[\partial^\mu \partial_\mu + m^2\right]\hat{\phi}(x) = 0, \tag{84}$$

$$\left[\partial^\mu \partial_\mu + m^2\right]\hat{\phi}^\dagger(x) = 0, \tag{85}$$

and keeping in mind that

$$\partial_{x_0}\Theta(x_0) \;=\; \delta(x_0), \tag{86}$$

one finds directly from the definition (1), the equations of motion of contour Green's function

$$\big(\Box_x + m^2\big)\Delta(x,y) \;=\; -\delta_C^{(4)}(x,y), \tag{87}$$

$$\big(\Box_y + m^2\big)\Delta(x,y) \;=\; -\delta_C^{(4)}(x,y), \tag{88}$$

where the contour Dirac delta is defined as

$$\delta_C^{(4)}(x,y) = \begin{cases} \delta^{(4)}(x-y) & \text{for } \; x_0, y_0 \; \text{ from the upper branch,} \\ 0 & \text{for } \; x_0, y_0 \; \text{ from the different branches,} \\ -\delta^{(4)}(x-y) & \text{for } \; x_0, y_0 \; \text{ from the lower branch.} \end{cases} \tag{89}$$

- Let us derive the equation of motion of the chronologically ordered Green's function which is, as we remember, defined as

$$i\Delta^c(x,y) \equiv \langle T^c \hat{\phi}(x)\hat{\phi}(y)\rangle. \tag{90}$$

The d'Alembertian acts only on the field operators of the Green function (90) but not on the density operator. So, one finds

$$\begin{aligned}\big(\Box_x + m^2\big)\Delta^c(x,y) \;=\;& -i\big\langle\big(\partial_{x_0}^2\Theta(x_0-y_0)\big)\hat{\phi}(x)\hat{\phi}(y) + \big(\partial_{x_0}^2\Theta(y_0-x_0)\big)\hat{\phi}(y)\hat{\phi}(x) \\ &+2\partial_{x_0}\Theta(x_0-y_0)\partial_{x_0}\hat{\phi}(x)\hat{\phi}(y) + 2\partial_{x_0}\Theta(y_0-x_0)\hat{\phi}(y)\partial_{x_0}\hat{\phi}(x) \\ &+\Theta(x_0-y_0)\big(\Box_x + m^2\big)\hat{\phi}(x)\hat{\phi}(y) + \Theta(y_0-x_0)\hat{\phi}(y)\big(\Box_x + m^2\big)\hat{\phi}(x)\big\rangle.\end{aligned} \tag{91}$$

Since the fields are assumed to obey the Klein-Gordon equations the terms from the last line in Eq. (91) vanish. Using the formula (86), we get

$$\begin{aligned}\big(\Box_x + m^2\big)\Delta^c(x,y) \;=\;& -i\big\langle\big(\partial_{x_0}\delta(x_0-y_0)\big)\hat{\phi}(x)\hat{\phi}(y) + \big(\partial_{x_0}\delta(y_0-x_0)\big)\hat{\phi}(y)\hat{\phi}(x) \\ &+2\delta(x_0-y_0)\partial_{x_0}\hat{\phi}(x)\hat{\phi}(y) + 2\delta(y_0-x_0)\hat{\phi}(y)\partial_{x_0}\hat{\phi}(x)\big\rangle.\end{aligned} \tag{92}$$

The expressions $\partial_{x_0}\delta(x_0-y_0)$ should be understood as

$$\int dx_0 \big(\partial_{x_0}\delta(x_0-y_0)\big) f(x) = -\int dx_0 \; \delta(x_0-y_0)\partial_{x_0} f(x), \tag{93}$$

where $f(x)$ is an arbitrary function and the partial integration has been performed. Therefore,

$$\begin{aligned}(\Box_x + m^2)\Delta^c(x,y) &= -i\langle -\delta(x_0-y_0)\big(\partial_{x_0}\hat{\phi}(x)\big)\hat{\phi}(y) + \delta(x_0-y_0)\hat{\phi}(y)\partial_{x_0}\hat{\phi}(x) \qquad (94)\\ &\quad +2\delta(x_0-y_0)\big(\partial_{x_0}\hat{\phi}(x)\big)\hat{\phi}(y) - 2\delta(x_0-y_0)\hat{\phi}(y)\partial_{x_0}\hat{\phi}(x)\rangle,\\ &= -i\langle \delta(x_0-y_0)\big(\partial_{x_0}\hat{\phi}(x)\big)\hat{\phi}(y) - \delta(x_0-y_0)\hat{\phi}(y)\partial_{x_0}\hat{\phi}(x)\rangle. \qquad (95)\end{aligned}$$

Remembering that in the canonical formalism the momentum conjugated to $\hat{\phi}(x)$ is

$$\hat{\pi}(x) = \dot{\hat{\phi}}(x) = \partial_{x_0}\hat{\phi}(x), \tag{96}$$

and using the equal-time commutation relation

$$[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{97}$$

one finally finds

$$(\Box_x + m^2)\Delta^c(x,y) = -\delta^{(4)}(x-y). \tag{98}$$

- The equations of motion of real-time Green's functions (6, 7, 8, 9) are immediately obtained from Eqs. (87, 88). The function $\Delta^>$ obeys the equations

$$\begin{aligned}\big(\Box_x + m^2\big)\Delta^>(x,y) &= 0, &(99)\\ \big(\Box_y + m^2\big)\Delta^>(x,y) &= 0. &(100)\end{aligned}$$

- Since a non-equilibrium system is, in general, inhomogeneous, the translational invariance is broken. Then, one usually uses the variables X and u defined by Eq. (36) and the Green's functions are written as

$$\Delta(x,y) = \Delta\big(X + \frac{1}{2}u, X - \frac{1}{2}u\big) \equiv \Delta(X,u).$$

- Using the variables X and u, Eqs. (99) and (100) become

$$\begin{aligned}\Big(\frac{1}{4}\partial_X^2 + \partial_X\partial_u + \partial_u^2 + m^2\Big)\Delta^>(X,u) &= 0, &(101)\\ \Big(\frac{1}{4}\partial_X^2 - \partial_X\partial_u + \partial_u^2 + m^2\Big)\Delta^>(X,u) &= 0. &(102)\end{aligned}$$

- Subtracting from each other and adding to each other Eqs. (101) and (102), we obtain

$$\begin{aligned}\partial_X\partial_u\Delta^>(X,u) &= 0, &(103)\\ \Big(\frac{1}{4}\partial_X^2 + \partial_u^2 + m^2\Big)\Delta^>(X,u) &= 0. &(104)\end{aligned}$$

- For the Wigner transformed function $\Delta^>(X,p)$ defined by Eq. (42), the equations (101) and (102) become

$$\begin{aligned}p_\mu\partial_X^\mu\Delta^>(X,p) &= 0, &(105)\\ \Big[\frac{1}{4}\partial_X^2 - p^2 + m^2\Big]\Delta^>(X,p) &= 0, &(106)\end{aligned}$$

which are known as the relativistic kinetic equation and mass-shell equation, respectively.

- We note that Eq. (105) written in a non-covariant notation has the familiar form of the transport equation
$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\Delta^{>}(X,p) = 0. \tag{107}$$
where the velocity $\mathbf{v}$ is $\mathbf{p}/p_0$.

- Eq. (106) shows that the Green function $\Delta^{>}(X,p)$ can be nonzero for the off-shell momenta, when $p^2 \neq m^2$.

- A macroscopic theory like a kinetic theory deals with system's characteristics averaged over scales larger than the particle Compton wavelength which is of the order of m^{-1}. So, one usually impose the *quasiparticle condition*
$$\Big|\frac{1}{m^2}\partial_X^2 \Delta^{>}(X,p)\Big| \ll |\Delta^{>}(X,p)|, \tag{108}$$
which says that $\Delta^{>}(X,p)$ weakly depends on X on the scale longer than the Compton wavelength.

- If the quasiparticle condition is satisfied, one can neglect the first term in Eq. (106) and then it becomes
$$\big[p^2 - m^2\big]\Delta^{>}(X,p) = 0, \tag{109}$$
which is solved by
$$\Delta^{>}(X,p) \sim \delta(p^2 - m^2). \tag{110}$$
Then, the function $\Delta^{>}(X,p)$ is nonzero only for the on-mass-shell four-momenta that is when $p^2 = m^2$.

- The same result (110) obviously holds for the function $\Delta^{<}(X,p)$. So, we write
$$i\Delta^{>}(X,p) = 2\pi\delta(p^2 - m^2)h(X,p), \tag{111}$$
$$i\Delta^{<}(X,p) = 2\pi\delta(p^2 - m^2)g(X,p), \tag{112}$$
where $h(X,p)$ and $g(X,p)$ are unknown functions.

- Since the difference $i\Delta^{>}(X,p) - i\Delta^{<}(X,p)$ equals the spectral function given by Eq. (72), one finds
$$\delta(p_0 - E_p) = \delta(p_0 - E_p)\big[h(X,p) - g(X,p)\big], \tag{113}$$
$$-\delta(p_0 + E_p) = \delta(p_0 + E_p)\big[h(X,p) - g(X,p)\big], \tag{114}$$
which are solved by
$$h(X,E_p,\mathbf{p}) = 1 + g(X,E_p,\mathbf{p}), \tag{115}$$
$$h(X,-E_p,\mathbf{p}) + 1 = g(X,-E_p,\mathbf{p}). \tag{116}$$

- Expressing the functions $h(X,p)$ and $g(X,p)$ through the $f(X,\mathbf{p})$ and $\bar{f}(X,\mathbf{p})$ as
$$h(X,p) = \Theta(p_0)\big[f(X,\mathbf{p}) + 1\big] + \Theta(-p_0)\bar{f}(X,-\mathbf{p}), \tag{117}$$
$$g(X,p) = \Theta(p_0)f(X,\mathbf{p}) + \Theta(-p_0)\big[\bar{f}(X,-\mathbf{p}) + 1\big], \tag{118}$$

the functions $\Delta^>(X,p)$ and $\Delta^<(X,p)$ acquire the form

$$\boxed{\Delta^>(X,p) = -\frac{i\pi}{E_p}\Big[\delta(E_p - p_0)\big(f(X,\mathbf{p})+1\big) + \delta(E_p + p_0)\bar{f}(X,-\mathbf{p})\Big],} \tag{119}$$

$$\boxed{\Delta^<(X,p) = -\frac{i\pi}{E_p}\Big[\delta(E_p - p_0)f(X,\mathbf{p}) + \delta(E_p + p_0)\big(\bar{f}(X,-\mathbf{p})+1\big)\Big],} \tag{120}$$

which strongly resembles that of Eqs. (69, 70). So, it is of no surprise that the functions $f(X,\mathbf{p})$ and $\bar{f}(X,-\mathbf{p})$ are the distribution functions of particles and antiparticles.

<u>Exercise</u>: Compute the current (46) and energy-momentum tensor (47) using the function (119).

- The function $\Delta^c(X,p)$ can be obtained from the relation

$$\Delta^c(x,y) = \Theta(x_0 - y_0)\Delta^>(x,y) + \Theta(y_0 - x_0)\Delta^<(x,y), \tag{121}$$

which leads to

$$\begin{aligned}\Delta^c(X,p) &= \int d^4u\, e^{ipu}\Theta(u_0)\int \frac{d^4k}{(2\pi)^4}e^{-iku}\Delta^>(X,k) \\ &+ \int d^4u\, e^{ipu}\Theta(-u_0)\int \frac{d^4k}{(2\pi)^4}e^{-iku}\Delta^<(X,k).\end{aligned} \tag{122}$$

Substituting the formulas (119, 120) into Eq. (122), one finds

$$\boxed{\Delta^c(X,p) = \frac{1}{p^2 - m^2 + i0^+} - \frac{i\pi}{E_p}\Big(\delta(p_0 - E_p)f(X,\mathbf{p}) + \delta(p_0 + E_p)\bar{f}(X,-\mathbf{p})\Big).} \tag{123}$$

We encounter here the integrals analogous to (75, 76) which are redefined by including the infinitesimal imaginary element $i0^+$ to make the integrals finite.

- The antichronologically ordered Green's function Δ^a is found from the relation

$$\Delta^a(x,y) = \Theta(x_0 - y_0)\Delta^<(x,y) + \Theta(y_0 - x_0)\Delta^>(x,y), \tag{124}$$

and the result is

$$\boxed{\Delta^a(X,p) = -\frac{1}{p^2 - m^2 - i0^+} - \frac{i\pi}{E_p}\Big(\delta(p_0 - E_p)f(X,\mathbf{p}) + \delta(p_0 + E_p)\bar{f}(X,-\mathbf{p})\Big).} \tag{125}$$

<u>Exercise</u>: Derive the formula (125).

Kubo-Martin-Schwinger condition

We close the Lecture with a derivation of an important general relation obeyed by the equilibrium Green's function which is known as the Kubo-Martin-Schwinger (KMS) condition.

- We derive the condition for the Green's functions $\Delta^<(x,y)$ and $\Delta^>(x,y)$ of complex scalar fields in equilibrium.
- As we remember, the Green's functions $\Delta^>(x,y)$ and $\Delta^<(x,y)$ are defined as

$$i\Delta^>(x,y) \equiv Z^{-1}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\hat{\phi}(x)\hat{\phi}^\dagger(y)\Big], \tag{126}$$

$$i\Delta^<(x,y) \equiv Z^{-1}\mathrm{Tr}\Big[e^{-\beta H}\hat{\phi}^\dagger(y)\hat{\phi}(x)\Big], \tag{127}$$

where $Z \equiv \mathrm{Tr}[e^{-\beta\hat{H}}]$.

- The Green's function (127) can be manipulated as

$$\begin{aligned} i\Delta^<(x,y) = Z^{-1}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\hat{\phi}^\dagger(y)\hat{\phi}(x)\Big] &= Z^{-1}\mathrm{Tr}\Big[\hat{\phi}(x)e^{-\beta\hat{H}}\hat{\phi}^\dagger(y)\Big] \\ &= Z^{-1}\mathrm{Tr}\Big[e^{-\beta\hat{H}}e^{\beta\hat{H}}\hat{\phi}(x)e^{-\beta\hat{H}}\hat{\phi}^\dagger(y)\Big]. \end{aligned} \tag{128}$$

- Writing down the four-positions x and y as $x = (t_1, \mathbf{x})$ and $y = (t_2, \mathbf{y})$, we observe that the fields $\phi(x)$ and $\phi(y)$ in the Heisenberg picture can be expressed as

$$\hat{\phi}(x) = \hat{\phi}(t_1,\mathbf{x}) = e^{i\hat{H}t_1}\hat{\phi}(\mathbf{x})e^{-i\hat{H}t_1}, \qquad \hat{\phi}(y) = \hat{\phi}(t_2,\mathbf{y}) = e^{i\hat{H}t_2}\hat{\phi}(\mathbf{y})e^{-i\hat{H}t_2}. \tag{129}$$

- Consequently,

$$e^{\beta\hat{H}}\hat{\phi}(t_1,\mathbf{x})e^{-\beta\hat{\hat{H}}} = \hat{\phi}(t_1 - i\beta, \mathbf{x}), \tag{130}$$

and Eq. (128) provides the relation

$$\Delta^<(x,y) = Z^{-1}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\hat{\phi}(t_1 - i\beta,\mathbf{x})\hat{\phi}^\dagger(y)\Big] = \Delta^>(t_1 - i\beta, \mathbf{x}, t_2, \mathbf{y}), \tag{131}$$

which is known as the Kubo-Martin-Schwinger condition.

- Since the equilibrium Green functions $\Delta^>(x,y)$ and $\Delta^<(x,y)$ depend on x and y only through $x-y$, we can put $y=0$ and the condition (131) becomes

$$\Delta^<(t,\mathbf{x}) = \Delta^>(t - i\beta, \mathbf{x}). \tag{132}$$

- The condition takes a more tangible form in the momentum space. Performing the Fourier transformation of Eq. (132), we get

$$\Delta^<(p) = \int_{-\infty}^{\infty} d^4x\; e^{ipx}\Delta^<(x) = \int_{-\infty}^{\infty} d^4x\; e^{ipx}\Delta^>(t - i\beta, \mathbf{x}). \tag{133}$$

- Changing the variable $t - i\beta \to t'$, one finds

$$\Delta^<(p) = e^{-\beta p_0}\int_{-\infty+i\beta}^{\infty+i\beta} dt' \int d^3x\; e^{ip_0t'}e^{-i\mathbf{p}\cdot\mathbf{x}}\Delta^>(t', \mathbf{x}). \tag{134}$$

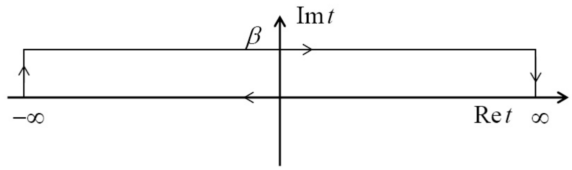

Figure 2: The contour in complex time plane

- Assuming that the function $\Delta^>(t',\mathbf{x})$ is analytic for $\beta \geq \mathrm{Im}t \geq 0$, that is, inside and on the contour shown in Fig.2, then, due to the Cauchy theorem, we get the equality

$$\int_{-\infty+i\beta}^{\infty+i\beta} dt\ e^{ip_0 t'} \Delta^>(t',\mathbf{x}) = \int_{-\infty}^{\infty} dt\ e^{ip_0 t'} \Delta^>(t',\mathbf{x}), \tag{135}$$

  as the integrals along the vertical parts of the contour vanish.

- The final form of the KMS condition is

$$\boxed{\Delta^<(p) = e^{-\beta p_0} \Delta^>(p).} \tag{136}$$