Real-time formalism cont.

In Lecture III we have started a discussion of the real-time or Keldysh-Schwinger formalism. We have defined various Green's functions and found their explicit form for non-interacting fields. Here we include an interaction into our discussion. Using the operator formalism of quantum field theory (QFT) we introduce the perturbative expansion of contour Green's functions which allows one to express the Green's functions of interacting fields through the sums of products of free Green's functions.

Perturbative expansion

- As we remember, the contour Green's function is defined as

$$i\Delta(x,y) \equiv \langle \tilde{T}\hat{\phi}(x)\hat{\phi}^\dagger(y)\rangle, \tag{1}$$

where

$$\langle \ldots \rangle \equiv \frac{\mathrm{Tr}[\hat{\rho}(t_0)\ldots]}{\mathrm{Tr}[\hat{\rho}(t_0)]}. \tag{2}$$

The trace is taken over a complete set of states of the system at an initial time t_0 and the operators and states are here in the Heisenberg picture.

- The key observation, which allows one to obtain the perturbative expansion, is that an operator $\hat{\mathcal{O}}_{\mathrm{H}}(t)$ in the Heisenberg picture can be written through the operator $\hat{\mathcal{O}}_{\mathrm{int}}(t)$ in the interaction picture as

$$\hat{\mathcal{O}}_{\mathrm{H}}(t) = \hat{U}_{\mathrm{int}}(t_0,t)\,\hat{\mathcal{O}}_{\mathrm{int}}(t)\,\hat{U}_{\mathrm{int}}(t,t_0), \tag{3}$$

where $\hat{U}_{\mathrm{int}}(t,t_0)$ is the evolution operator in the interaction picture which is

$$\begin{aligned}\hat{U}_{\mathrm{int}}(t,t_0) &= T^c e^{-i\int_{t_0}^{t} dt'\,\hat{H}^I_{\mathrm{int}}(t')} \qquad (4)\\ &= 1 - i\int_{t_0}^{t} dt'\,\hat{H}^I_{\mathrm{int}}(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T^c \hat{H}^I_{\mathrm{int}}(t_1)\,\hat{H}^I_{\mathrm{int}}(t_2) + \ldots\end{aligned}$$

$\hat{H}^I_{\mathrm{int}}$ is the interaction Hamiltonian in the interaction picture.

Exercise: Prove the equality (3).

- Choosing $t > t_0$, the expectation value of the operator $\mathcal{O}_{\mathrm{H}}(t)$ is expressed as

$$\begin{aligned}\langle \hat{\mathcal{O}}_{\mathrm{H}}(t)\rangle &= \langle \hat{U}_{\mathrm{int}}(t_0,t)\,\hat{\mathcal{O}}_{\mathrm{int}}(t)\,\hat{U}_{\mathrm{int}}(t,t_0)\rangle\\ &= \Big\langle T^a\big[e^{-i\int_{t}^{t_0} dt'\,\hat{H}^I_{\mathrm{int}}(t')}\big]\,\mathcal{O}_{\mathrm{int}}(t)\,T^c\big[e^{-i\int_{t_0}^{t} dt'\,\hat{H}^I_{\mathrm{int}}(t')}\big]\Big\rangle, \qquad (5)\end{aligned}$$

where T^c and T^a take care of chronologization and antichronologization of the operators within the square brackets.

- The expression (5) can be rewritten as

$$\langle \hat{\mathcal{O}}_{\mathrm{H}}(t)\rangle = \big\langle \tilde{T}\big[e^{-i\int_C dt'\,\hat{H}^I_{\mathrm{int}}(t')}\,\hat{\mathcal{O}}_{\mathrm{int}}(t)\,\big]\big\rangle = \frac{\mathrm{Tr}\big[\hat{\rho}(t_0)\tilde{T}[e^{-i\int_C dt'\,\hat{H}^I_{\mathrm{int}}(t')}\hat{\mathcal{O}}_{\mathrm{int}}(t)]\big]}{\mathrm{Tr}\big[\hat{\rho}(t_0)\big]}. \tag{6}$$

The time integral (denoted as $\int_C$) is taken from t_0 to t_0 along the Keldysh contour shown in Fig. 1, where $t_{\mathrm{max}} > t > t_0$, and $\tilde{T}$ orders the operators along the contour.

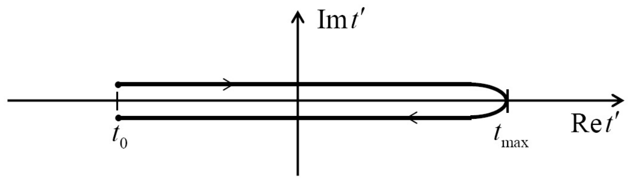

Figure 1: The Keldysh contour

- Generalizing the formula (6), the contour Green's function (1) is written as

$$i\Delta(x,y) = \Big\langle \tilde{T}\big[e^{-i\int_C dt\, \hat{H}^I_{\rm int}(t)}\, \hat{\phi}_{\rm int}(x)\, \hat{\phi}^\dagger_{\rm int}(y)\big]\Big\rangle, \tag{7}$$

  where $t_{\max}$ is bigger than x_0 and y_0, and both x_0 and y_0 are bigger than t_0. This conditions are automatically satisfied when t_0 is shifted to $-\infty$ and $t_{\max}$ to ∞.

- Expanding the exponential function in the formula (7) as in Eq. (4), we get the perturbative expansion of contour Green's function.

- One may worry that the operators in Eq. (7) are in the interaction picture while the states are in the Heisenberg picture. However, when the exponential function is expanded and the Wick theorem, which is discussed further on, is applied, the contour Green's function (7) is expressed as a sum of products of free Green's functions. And when we deal with free fields the Heisenberg and interaction pictures coincide with each other.

- The problem to be solved is how to calculate the expressions which originate from the perturbative expansion of the formula (7) which are of the form

$$\Big\langle \tilde{T}\big[\hat{H}^I_{\rm int}(t_1)\, \hat{H}^I_{\rm int}(t_2)\, \dots \hat{H}^I_{\rm int}(t_n)\, \hat{\phi}_{\rm int}(x)\, \hat{\phi}^\dagger_{\rm int}(y)\big]\Big\rangle, \tag{8}$$

  where n numerates the terms in the expansion (4).

- Further on, we suppress the subscript 'int'.

- The Lagrangian density of interacting fields includes, except the terms quadratic in fields, the term or terms of higher power. As an example, we consider the interacting scalar complex field of the Lagrangian density

$$\hat{\mathcal{L}}(x) = \partial^\mu\hat{\phi}(x)\partial_\mu\hat{\phi}^\dagger(x) - m^2\hat{\phi}(x)\hat{\phi}^\dagger(x) - \frac{\lambda}{2!\,2!}\big(\hat{\phi}(x)\hat{\phi}^\dagger(x)\big)^2, \tag{9}$$

  where the last quartic term represents the interaction and λ is the coupling constant.

- As we remember, there is a conserved charge in the theory described by the Lagrangian (9) and there are positively and negatively charged particles.

- The interaction Hamiltonian equals

$$\hat{H}^I(t) = \frac{\lambda}{2!\,2!}\int d^3x\, \big(\hat{\phi}(x)\hat{\phi}^\dagger(x)\big)^2, \tag{10}$$

  where $x = (t, \mathbf{x})$.

- The perturbative expansion is constructed out of non-interacting fields which obey Klein-Gordon equations. Therefore, the fields are expressed as superpositions of plane waves

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \left[e^{-ikx}\hat{a}(\mathbf{k}) + e^{ikx}\,\hat{b}^\dagger(\mathbf{k})\right], \tag{11}$$

$$\hat{\phi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \left[e^{-ikx}\,\hat{b}(\mathbf{k}) + e^{ikx}\hat{a}^\dagger(\mathbf{k})\right]. \tag{12}$$

  where $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$ with $\omega_{\mathbf{k}} \equiv \sqrt{m^2 + \mathbf{k}^2}$, and $\hat{a}^\dagger(\mathbf{k})$, $\hat{b}^\dagger(\mathbf{k})$ are creation operators of positively and negatively charged particles, respectively, and $\hat{a}(\mathbf{k})$, $\hat{b}(\mathbf{k})$ are annihilation operators.

- We see that the expression (8) is the expectation value of a product of field operators ordered along the Keldysh contour. The fields are expressed through the creation and annihilation operators due to Eqs. (11, 12). So, the expression to be computed is

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle, \tag{13}$$

  where α_i is either the annihilation or creation operator.

Wick's theorem

- The Wick's theorem states that the expectation value $\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle$ can be expressed as a sum of products of $\langle \hat{\alpha}_i \hat{\alpha}_j \rangle$.

- We are going to compute $\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle$. Since the trace is taken over the eigenstates of particle number operator, the numbers of creation and annihilation operators of a given particle species must be the same in $\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle$. So, the number m must be even. Otherwise the expectation value vanishes.

- We manipulate the expression $\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle$ moving the operator $\hat{\alpha}_1$ from left to right. In the first step we get

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle = \langle [\hat{\alpha}_1, \hat{\alpha}_2] \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle + \langle \hat{\alpha}_2 \hat{\alpha}_1 \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle, \tag{14}$$

  where $[\hat{\alpha}_1, \hat{\alpha}_2]$ is the commutator of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ which is assumed to be a $c-$number. Therefore, it can be pull-out of the expectation value. So, we have

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle = [\hat{\alpha}_1, \hat{\alpha}_2] \langle \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle + \langle \hat{\alpha}_2 \hat{\alpha}_1 \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle. \tag{15}$$

- Moving α_1 further to the right we commute it with α_3, α_4, etc. Thus, we obtain

$$\begin{aligned} \langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle &= [\hat{\alpha}_1, \hat{\alpha}_2] \langle \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle + [\hat{\alpha}_1, \hat{\alpha}_3] \langle \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle + \dots + [\hat{\alpha}_1, \hat{\alpha}_m] \langle \hat{\alpha}_2 \dots \hat{\alpha}_{m-1} \rangle \\ &\quad + \langle \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_m \hat{\alpha}_1 \rangle. \end{aligned} \tag{16}$$

- Now, we write down the last term explicitly and use the cyclic property of trace

$$\langle \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_m \hat{\alpha}_1 \rangle \equiv \frac{\mathrm{Tr}[\hat{\rho}\,\hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_m \hat{\alpha}_1]}{\mathrm{Tr}[\hat{\rho}]} = \frac{\mathrm{Tr}[\hat{\alpha}_1 \hat{\rho}\,\hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_m]}{\mathrm{Tr}[\hat{\rho}]}. \tag{17}$$

- In the next step, we make an important assumption about the density matrix $\hat{\rho}$ which describes the initial state. Specifically, we assume that

$$\hat{\alpha}_i \hat{\rho} = \hat{\rho}\, \hat{\alpha}_i\, \eta_i \tag{18}$$

where η^i is a $c-$number. The assumption (18) limits a class of initial states which allow for the Wick's decomposition. As we will see, it is satisfied by the equilibrium density matrix $\hat{\rho} = e^{-\beta \hat{H}}$ and by matrices of the form $\hat{\rho} = e^{\hat{\mathcal{A}}}$ where the operator $\hat{\mathcal{A}}$ linearly depends on the particle number operator.

- Using the equality (18), Eq. (17) provides

$$\langle \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_m \hat{\alpha}_1 \rangle = \eta_1 \langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle. \tag{19}$$

- Substituting the result (19) into Eq. (16), we get

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle = \frac{[\hat{\alpha}_1, \hat{\alpha}_2]}{1-\eta_1} \langle \hat{\alpha}_3 \dots \hat{\alpha}_m \rangle + \frac{[\hat{\alpha}_1, \hat{\alpha}_3]}{1-\eta_1} \langle \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle + \dots + \frac{[\hat{\alpha}_1, \hat{\alpha}_m]}{1-\eta_1} \langle \hat{\alpha}_2 \dots \hat{\alpha}_{m-1} \rangle. \tag{20}$$

- In the l.h.s. of Eq. (20) there is the expectation value of the product of m operators while in the r.h.s. of the equation there is a sum of the expectation values of the products of $(m-2)$ operators. Applying repeatedly the same procedure to every expectation value from Eq. (20), we finally obtain

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle = \sum_{\text{permutations}} \frac{[\hat{\alpha}_1, \hat{\alpha}_2]}{1-\eta_1} \frac{[\hat{\alpha}_3, \hat{\alpha}_4]}{1-\eta_3} \dots \frac{[\hat{\alpha}_{m-1}, \hat{\alpha}_m]}{1-\eta_{m-1}}, \tag{21}$$

where the sum is over all permutations of pairs (contractions) of the operators $\hat{\alpha}_i$ and $\hat{\alpha}_j$.

- Since

$$\langle \hat{\alpha}_i \hat{\alpha}_j \rangle = \frac{[\hat{\alpha}_i, \hat{\alpha}_j]}{1-\eta_i}, \tag{22}$$

we see that the equality (21) can be rewritten as

$$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m \rangle = \sum_{\text{permutations}} \langle \hat{\alpha}_1 \hat{\alpha}_2 \rangle \langle \hat{\alpha}_3 \hat{\alpha}_4 \rangle \langle \hat{\alpha}_{m-1} \hat{\alpha}_m \rangle, \tag{23}$$

which shows that the expectation value of the product of m operators ($m > 2$) can be expressed as a sum of products of expectation values of products of two operators. This is the statement of Wick's theorem.

- We note that a big part of terms in Eq. (23), which are obtained by permutation of the operators $\hat{\alpha}_1$, $\hat{\alpha}_2, \dots \hat{\alpha}_m$ vanish. A non-zero contribution is provided only by the terms where all two-operator expectation values include one creation and one annihilation operator of the same particle species.

Simple examples

- To see how the Wick's theorem works, let us first consider the expectation value $\langle \hat{a}_i^\dagger \hat{a}_j \rangle$, where $\hat{a}_i^\dagger$ and $\hat{a}_j$ are the creation and annihilation operators of the discretized model of real scalar field which is discussed in detail in Lecture I. The field is in thermodynamic equilibrium and the density matrix is $\hat{\rho} = e^{-\beta \hat{H}}$. So, we are going to consider the following expression

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle = \frac{\mathrm{Tr}[e^{-\beta \hat{H}} \hat{a}_i^\dagger \hat{a}_j]}{\mathrm{Tr}[e^{-\beta \hat{H}}]}. \tag{24}$$

- Since the trace is computed with the eigenstates of the particle number operator $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$, one immediately realizes that the expression (24) is nonzero only for $i = j$. Then, one finds

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle = \delta^{ij} \frac{\mathrm{Tr}[e^{-\beta \hat{H}} \hat{n}_i]}{\mathrm{Tr}[e^{-\beta \hat{H}}]} = \delta^{ij} n_i. \tag{25}$$

- On the other hand Eq. (22) tell us that

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle = \frac{[\hat{a}_i^\dagger, \hat{a}_j]}{1 - \eta_i}. \tag{26}$$

- We are going to show that the r.h.s of Eq. (25) equals the the r.h.s of Eq. (26).
- Since the annihilation and creation operators satisfy the commutation relation

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta^{ij}, \tag{27}$$

see Lecture I, Eq. (26) becomes

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle = \frac{\delta^{ij}}{\eta_i - 1}. \tag{28}$$

- So, we have to find η_i which is defined through Eq. (18). For $\alpha_i = \hat{a}_i^\dagger$ and $\hat{\rho} = e^{-\beta \hat{H}}$, Eq. (18) is

$$\hat{a}_i^\dagger e^{-\beta \hat{H}} = e^{-\beta \hat{H}} \hat{a}_i^\dagger \eta_i, \tag{29}$$

where $\hat{H} = \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j$.

- Expanding the exponential function, one commutes $\hat{a}_i^\dagger$ with $\hat{\rho}$ and finds after a rather tedious analysis that

$$\hat{a}_i^\dagger e^{-\beta \hat{H}} = e^{-\beta \hat{H}} \hat{a}_i^\dagger + e^{-\beta \hat{H}} \left(e^{\beta \omega_i} - 1\right) \hat{a}_i^\dagger = e^{-\beta \hat{H}} \hat{a}_i^\dagger \, e^{\beta \omega_i}. \tag{30}$$

Therefore,

$$\eta_i = e^{\beta \omega_i} = \frac{n_i + 1}{n_i}, \tag{31}$$

where

$$n_i = \frac{1}{e^{\beta \omega_i} - 1}. \tag{32}$$

<u>Exercise</u>: Prove the equality (29) with $\eta_i = e^{\beta \omega_i}$.

- Substituting the result (31) into Eq. (28), we reproduce Eq. (25).
- Let us now consider $\langle \hat{a}_i \hat{a}_j^\dagger \rangle$. Analogously to Eq. (25), one finds

$$\langle \hat{a}_i \hat{a}_j^\dagger \rangle = \frac{\mathrm{Tr}[e^{-\beta \hat{H}} (\delta^{ij} + \delta^{ij} \hat{n}_i)]}{\mathrm{Tr}[e^{-\beta \hat{H}}]} = \delta^{ij} (n_i + 1). \tag{33}$$

- The analog of Eq. (28) is

$$\langle \hat{a}_i \hat{a}_j^\dagger \rangle = \frac{\delta^{ij}}{1 - \eta_i}, \tag{34}$$

and η_i in the case equals

$$\eta_i = e^{-\beta\omega_i} = \frac{n_i}{n_i + 1}. \tag{35}$$

Exercise: Derive Eq. (35) from Eq. (18) for $\alpha_i = \hat{a}_i$ and $\hat{\rho} = e^{-\beta\hat{H}}$.

Feynman rules

A calculation of successive terms of perturbative expansion of the contour Green's function can be changed into an algorithmic procedure with a set of mnemonic rules analogous to the Feynman rules of the vacuum QFT. The key element of the procedure are the Feynman diagrams. We are not going to elaborate the Feynman rule which depend on the field theory under consideration. Since the Feynman rules of the Keldysh-Schwinger formalism are rather similar to those of the vacuum QFT, we discuss here only the differences.

- In the vacuum QFT this is the Feynman, or time-ordered, propagator which is perturbatively expanded. In the Keldysh-Schwinger formalism the contour Green's function is expanded. As we know, the contour Green's function comprises four Green's functions of real time arguments and it is often a non-trivial task to extract the function of interest, say $\Delta^{\gtrless}$, form the contour function.
- The second important difference is that instead of the time integration from $-\infty$ to ∞, we integrate along the Keldysh contour with t_0 shifted to $-\infty$ and $t_{\rm max}$ to ∞.
- The third difference is a role the tadpoles *i.e.* the loops formed by single lines as that one shown in Fig. 2. A tadpole corresponds to the Green's function of space-time arguments equal to each other like $\Delta(x, x)$. The tadpoles do not appear in the vacuum QFT as long the field vacuum expectation values vanish. In the operator formalism they are effectively eliminated by the operator normal ordering of the interaction Hamiltonian H^I. Then, $\langle 0| : \hat{\phi}(x)\, \hat{\phi}^\dagger(x) : |0\rangle = 0$. In the path-integral formulation the tadpoles, which are actually infinite, show up but are canceled by properly chosen counterterms.
- In statistical QFT the tadpoles play an important physical role. They appear due finite system's density. Since the function $\Delta(x, x)$ is ill defined (the contour ordering does not work), the tadpole is represented by $\Delta^<(x, x)$.
- If the system under study is stationary and homogeneous, that is space-time translationally invariant, as it typically happens in case of equilibrium systems, the perturbative expansion in momentum space in statistical QFT is fully analogous to that in vacuum QFT. The situation is much more complex when we deal with non-equilibrium inhomogeneous systems. Then, the Green's functions depend not only on the difference of their space-time arguments but on both arguments. Instead of the Fourier transformation, one has to use the Wigner transformation which leads to some complications.

The differences will be illustrated with a few examples discussed below.

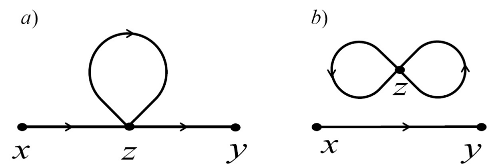

Figure 2: The first order contributions to the contour Green's function

First-order contributions to Δ

- Let us now discuss the first-order correction to the contour Green's function in a theory of self-interacting complex scalar field described by the Lagrangian (9). The expression to be computed is

$$i\Delta^{(1)}(x,y) = -\frac{i\lambda}{2!\,2!}\int_C d^4z \langle \tilde{T}\big[\big(\hat{\phi}(z)\hat{\phi}^\dagger(z)\big)^2 \hat{\phi}(x)\hat{\phi}^\dagger(y)\big]\rangle, \tag{36}$$

  where the integral over z_0 is performed along the Keldysh contour shown in Fig. 1.

- The plane-wave decompositions (11, 12) show that the expectation values like $\langle \tilde{T}\hat{\phi}(z)\hat{\phi}(x)\rangle$ and $\langle \tilde{T}\hat{\phi}(z)^\dagger\hat{\phi}^\dagger(y)\rangle$ vanish. So, one realizes that there are two contributions

$$i\Delta_a^{(1)}(x,y) = -i\lambda\int_C d^4z \langle \tilde{T}[\hat{\phi}(z)\hat{\phi}^\dagger(y)]\rangle\langle \tilde{T}[\hat{\phi}(z)\hat{\phi}^\dagger(z)]\rangle\langle \tilde{T}[\hat{\phi}^\dagger(z)\hat{\phi}(x)]\rangle, \tag{37}$$

$$i\Delta_b^{(1)}(x,y) = -\frac{i\lambda}{2}\langle \tilde{T}[\hat{\phi}(x)\hat{\phi}^\dagger(y)]\rangle\int_C d^4z \langle \tilde{T}[\hat{\phi}(z)\hat{\phi}^\dagger(z)]\rangle\langle \tilde{T}[\hat{\phi}(z)\hat{\phi}^\dagger(z)]\rangle. \tag{38}$$

  In case of the contribution a, there are two operators $\hat{\phi}(z)$ to be paired with $\hat{\phi}^\dagger(y)$ and two operators $\hat{\phi}^\dagger(z)$ to be paired with $\hat{\phi}(z)$, the combinatorial factor of 4 is included in the formula (37). In case of the contribution b, the combinatorial factor equals 2.

- The results (37, 38) can be written as

$$\Delta_a^{(1)}(x,y) = i\lambda\int_C d^4z\, \Delta(x,z)\,\Delta(z,z)\,\Delta(z,y), \tag{39}$$

$$\Delta_b^{(1)}(x,y) = \frac{i\lambda}{2}\Delta(x,y)\int_C d^4z\, \Delta(z,z)\,\Delta(z,z), \tag{40}$$

  where $\Delta(x,y)$ is the free contour Green's function. The two contributions are represented graphically by two diagrams shown in Fig. 2.

Second-order contributions to Δ

- The expression to be computed is

$$i\Delta^{(2)}(x,y) = -\frac{\lambda^2}{(2!\,2!)^2 2!}\int_C d^4z_1\int_C d^4z_2 \langle \tilde{T}\big[\big(\hat{\phi}(z_1)\hat{\phi}^\dagger(z_1)\big)^2\big(\hat{\phi}(z_2)\hat{\phi}^\dagger(z_2)\big)^2\hat{\phi}(x)\hat{\phi}^\dagger(y)\big]\rangle. \tag{41}$$

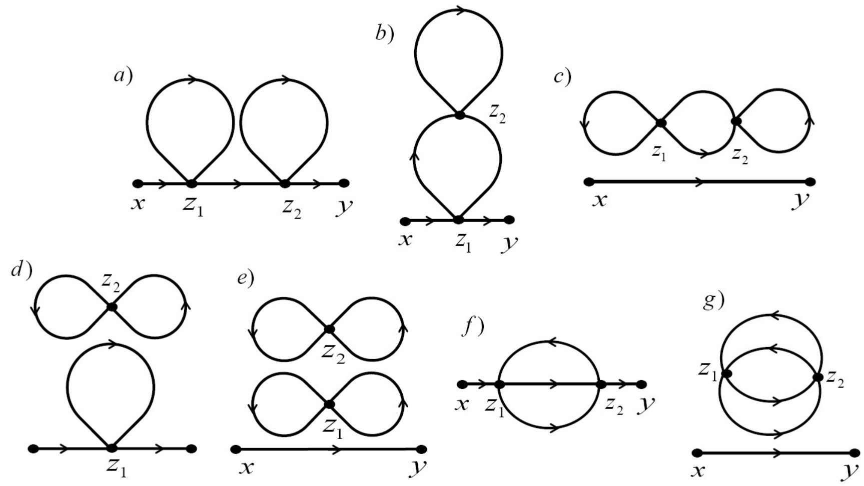

Figure 3: The second order contributions to the contour Green's function

- There are seven second-order contributions

$$\Delta_a^{(2)}(x,y) = -\#\lambda^2 \int_C d^4z_1 \int_C d^4z_2\, \Delta(x,z_1)\,\Delta(z_1,z_1)\,\Delta(z_1,z_2)\,\Delta(z_2,z_2)\,\Delta(z_2,y), \quad (42)$$

$$\Delta_b^{(2)}(x,y) = -\#\lambda^2 \int_C d^4z_1 \int_C d^4z_2\, \Delta(x,z_1)\,\Delta(z_1,z_2)\,\Delta(z_2,z_1)\,\Delta(z_2,z_2)\,\Delta(z_1,y), \quad (43)$$

$$\Delta_c^{(2)}(x,y) = -\#\lambda^2 \Delta(x,y) \int_C d^4z_1 \int_C d^4z_2 \Delta(z_1,z_1)\,\Delta(z_1,z_2)\,\Delta(z_2,z_1)\,\Delta(z_2,z_2), \quad (44)$$

$$\Delta_d^{(2)}(x,y) = -\#\lambda^2 \int_C d^4z_1\, \Delta(x,z_1)\,\Delta(z_1,z_1)\,\Delta(z_1,y) \int_C d^4z_2\, \Delta(z_2,z_2)\,\Delta(z_2,z_2), \quad (45)$$

$$\Delta_e^{(2)}(x,y) = -\#\lambda^2 \Delta(x,y) \int_C d^4z_1\, \Delta(z_1,z_1)\,\Delta(z_1,z_1) \int_C d^4z_2\, \Delta(z_2,z_2)\,\Delta(z_2,z_2), \quad (46)$$

$$\Delta_f^{(2)}(x,y) = -\#\lambda^2 \int_C d^4z_1 \int_C d^4z_2\, \Delta(x,z_1)\,\Delta(z_1,z_2)\,\Delta(z_1,z_2)\,\Delta(z_2,z_1)\,\Delta(z_2,y), \quad (47)$$

$$\Delta_g^{(2)}(x,y) = -\#\lambda^2 \Delta(x,y) \int_C d^4z_1 \int_C d^4z_2 \Delta(z_1,z_2)\,\Delta(z_1,z_2)\,\Delta(z_2,z_1)\,\Delta(z_2,z_1), \quad (48)$$

where # denotes a numerical coefficient. The contributions are represented by the diagrams shown in Fig. 3.

Exercise: Derive the numerical factors in Eqs. (42 - 47).

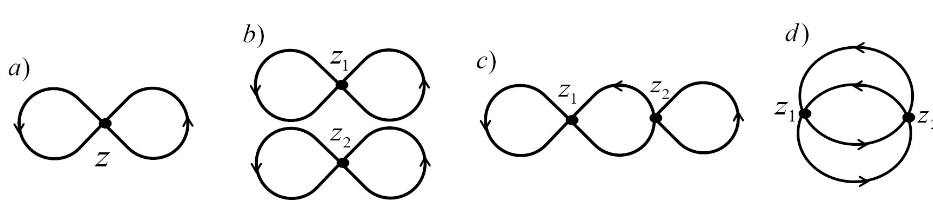

Figure 4: The first and second order contributions to the quantity (49)

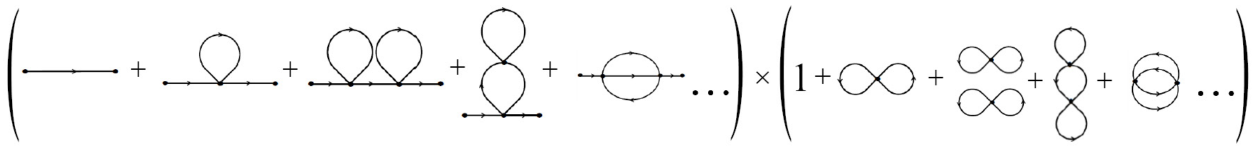

Figure 5: The perturbative expansion of the contour Green's function

Connected and disconnected diagrams

- The diagram b) in Fig. 2 and the diagrams c), d) and e) in Fig. 3 can be split each into into two diagrams not cutting any line. These are the disconnected diagrams.

- One observes that the perturbative expansion of the quantity

$$\frac{\mathrm{Tr}\big[\hat{\rho}(t_0)\,\tilde{T}[e^{-i\int_C dt'\,\hat{H}^I_{\rm int}(t')}]\big]}{\mathrm{Tr}[\hat{\rho}(t_0)]} \tag{49}$$

  gives unity in the zeroth order, the diagram a) from Fig. 4 in the first order and the diagrams b) and c) from Fig. 4 in the second order.

- It is also not difficult to observe that the perturbative expansion of the contour Green's function can be expressed in a way which is shown graphically in Fig. 5.

- Combing the observations, one finds that the disconnected diagrams can be eliminated from the perturbative expansion of contour Green's function by redefinition of the function as

$$i\Delta(x,y) \equiv \frac{\mathrm{Tr}\big[\hat{\rho}(t_0)\tilde{T}[e^{-i\int_C dt\,\hat{H}^I_{\rm int}(t)}\,\hat{\phi}_{\rm int}(x)\,\hat{\phi}^\dagger_{\rm int}(y)]\big]}{\mathrm{Tr}[\hat{\rho}(t_0)]} \tag{50}$$

$$\longrightarrow \quad i\Delta(x,y) \equiv \frac{\mathrm{Tr}\big[\hat{\rho}(t_0)\tilde{T}[e^{-i\int_C dt\,\hat{H}^I_{\rm int}(t)}\,\hat{\phi}_{\rm int}(x)\,\hat{\phi}^\dagger_{\rm int}(y)]\big]}{\mathrm{Tr}\big[\hat{\rho}(t_0)\tilde{T}[e^{-i\int_C dt\,\hat{H}^I_{\rm int}(t)}]\big]} \tag{51}$$

  As one see, the exponential $\tilde{T}[e^{-i\int_C dt\,\hat{H}^I_{\rm int}(t)}]$ is included in the denominator of the definition of the contour Green's function.

- Further on, we will use the redefined Green's functions.

Perturbative expansion of real time Green's functions

- As already mentioned, this is the contour Green's function which is perturbatively expanded in the Keldysh-Schwinger formalism. Let us now discuss how to extract the real-time Green's functions from the contour one. As an example we consider the first order contribution (40).

- If we want to get the contribution to, say, the function $\Delta^>(x,y)$ we put x_0 on the lower branch of the contour and y_0 on the upper one. We split the integration over z in the formula (40) into the integration from $-\infty$ to ∞ and from ∞ to $-\infty$. In the first part z_0 belongs to the upper branch of the contour and in the second part to the lower branch.

Thus, we get

$$\begin{aligned}
\Delta^{>}_{(1)}(x,y) &= i\lambda \int_{-\infty}^{\infty} dz_0 \int d^3z\, \Delta^{>}(x,z)\, \Delta^{>}(z,z)\, \Delta^{c}(z,y) \\
&\quad -i\lambda \int_{\infty}^{-\infty} dz_0 \int d^3z\, \Delta^{a}(x,z)\, \Delta^{>}(z,z)\, \Delta^{>}(z,y) \\
&= i\lambda \int d^4z \Big[\Delta^{>}(x,z)\, \Delta^{c}(z,y) - \Delta^{a}(x,z)\, \Delta^{>}(z,y)\Big] \Delta^{>}(z,z).
\end{aligned} \tag{52}$$

- In analogous way one can find the expressions for $\Delta^{>}_{(1)}(x,y)$, $\Delta^{c}_{(1)}(x,y)$ and $\Delta^{a}_{(1)}(x,y)$.
- At higher orders when there are several integrals along the contour, the procedure to extract a real-time Green's function of interest from the contour Green's function becomes rather tedious. There is a graphical method to do it which will be discussed later on.

Green's functions in momentum dependence

- When we deal with a system which is stationary and homogeneous, as is usually the case of equilibrium systems, the perturbative expansion in momentum space is obtained in statistical QFT analogously to that in vacuum QFT. As long as the system of interest is space-time translationally invariant, the Green's functions depend only the difference of their space-time arguments, and thus we can apply the Fourier transformation to get the Green's functions in momentum space.
- As an example we consider the first order contribution to $\Delta^{>}(x,y)$ given by Eq. (52). When the system is translationally invariant, Eq. (52) becomes

$$\Delta^{>}_{(1)}(x) = i\lambda \int d^4z \Big[\Delta^{>}(x-z)\, \Delta^{c}(z) - \Delta^{a}(x-z)\, \Delta^{>}(z)\Big] \Delta^{>}(0), \tag{53}$$

where we put $y = 0$.

- Performing the Fourier transformation, Eq. (53) gives

$$\Delta^{>}_{(1)}(p) = i\lambda \Big[\Delta^{>}(p)\, \Delta^{c}(p) - \Delta^{a}(p)\, \Delta^{>}(p)\Big] \int \frac{d^4k}{(2\pi)^4}\, \Delta^{>}(k). \tag{54}$$

- When the system under study is time dependent and/or inhomogeneous, that is, it is not translationally invariant, the Green's function depend on both spatial arguments. Then, instead of the Fourier transformation one can apply the Wigner transformation to describe the system in momentum space.
- The Wigner transformation is defined as

$$\Delta(X,p) = \int d^4u\, e^{ip\cdot u} \Delta\Big(X + \frac{1}{2}\,u, X - \frac{1}{2}\,u\Big), \tag{55}$$

and its inverse is

$$\Delta\Big(X + \frac{1}{2}\,u, X - \frac{1}{2}\,u\Big) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot u} \Delta(X,p). \tag{56}$$

- Applying the Wigner transformation to Eq. (52), we get

$$\Delta^{>}_{(1)}(X,p) = i\lambda \int d^4u \int d^4z \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \tag{57}$$
$$\times \exp\Big[ip\cdot u - iq_1 \cdot \Big(X + \frac{1}{2}u - z\Big) - iq_2 \cdot \Big(z - X + \frac{1}{2}u\Big)\Big]$$
$$\times \Big[\Delta^{>}\Big(\frac{1}{2}X + \frac{1}{4}u + \frac{1}{2}z, q_1\Big)\,\Delta^{c}\Big(\frac{1}{2}X - \frac{1}{4}u + \frac{1}{2}, q_2\Big)$$
$$- \Delta^{a}\Big(\frac{1}{2}X + \frac{1}{4}u + \frac{1}{2}z, q_1\Big)\,\Delta^{>}\Big(\frac{1}{2}X - \frac{1}{4}u + \frac{1}{2}, q_2\Big)\Big]\Delta^{>}(z,p).$$

  One observes that the integrals over u and z, which in case of translationally invariant system produced the delta functions and allowed one to greatly simplify the expression, are no longer trivial. The result (57) can be can be simplified only under extra assumptions on the Green's functions.

- As we can see, the transition from the description of a given system in the coordinate space to the description in the momentum space is simple in the case of translationally invariant systems. In case of systems, which are not translationally invariant, the situation is much more complex.